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THE GEOMETRY OF THE CIRCLE

AND

MATHEMATICS

AS APPLIED TO GEOMETRY BY MATHEMATICIANS,
SHEWN TO BE

A MOCKERY, DELUSION, AND A SNARE.

LETTER

TO

G. G. STOKES, Esq., M.A., F.R.S., D.C.L.

President Elect of the British Association for the Advancement of Science, 1869—70.

By JAMES SMITH, Esq.,

MEMBER OF THE MERSEY DOCKS AND HARBOUR BOARD, AND EX-CHAIRMAN OF THE
LIVERPOOL LOCAL MARINE BOARD.

BIBLIOTHECA BODLEIANA

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LONDON SIMPKIN MARSHALL & CO. STATIONERS' HALL COURT.
1869.

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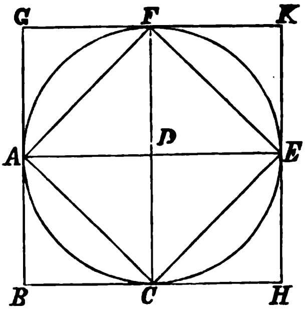
183. e. 22.

TO THE READER.

I ASSUME that you are a Geometer. If so, you may have mastered every proposition of Euclid, without having acquired even so much knowledge of Arithmetic, as to be master of the Multiplication Table.* But, if you are a Mathematician, you will *know* that, we cannot apply Mathematics to pure Geometry, without the aid of "*that indispensable instrument of science, Arithmetic,*" upon which all Mathematics are founded.



In the geometrical figure in the margin, A B C D is a square on the radius of the circle; and, A C E F is a square on A C, a diagonal of the square A B C D, by construction. It is not only axiomatic, but self-evident, that A C E F is an inscribed square, and G B H K a circumscribing square to the circle.

As an Arithmetician you will *know* that, the mean proportional between any given whole number and its fourth part, is equal to half the given number, or the double of its fourth part; and it follows of necessity, that the mean proportional between the area of a circumscribing square to any circle, and the area of a square on the radius of the circle, is equal to the area of an inscribed square to the circle.

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* I have the authority of Mr. J. Radford Young for this statement, and he is a living "*recognised Mathematician.*" (See, Men of the Time, and Mr. J. R. Young's Treatise on Algebra, in the well-known scientific work, *Orr's Circle of the Sciences*).

Now, the semi-radius of any circle is equal to one-fourth part of its diameter; and by parity of reasoning, the mean proportional between the diameter and semi-radius of any circle, is equal to the radius of the circle: or in other words, the mean proportional between the diameter and semi-radius of any circle, is equal to half the diameter or twice the semi-radius of the circle; and this is unquestionably true, whether the diameter of the circle be represented by any whole number, by the square root of any whole number, or by the square root of any square number.

For example:

Let F C, the diameter of the circle, be represented by the arithmetical quantity 32.

Then:

32 is a whole number, but not a square number. But, $\frac{32}{2} = 16 =$ radius of the circle: and, $\frac{32}{4} = 8$ $=$ semi-radius: therefore, the mean proportional between the diameter and semi-radius of the circle $= \sqrt{32 \times 8}$ $= \sqrt{256} = 16$, and is equal to half the diameter of the circle, or twice the semi-radius; that is to say, is equal to the radius of the circle.

Again:

Let F C, the diameter of the circle, be represented by the definite arithmetical expression $\sqrt{25}$.

Then:

25 is not only a whole number, but is also a square

number. Hence: $\frac{\sqrt{25}}{2} = \sqrt{\frac{25}{2^2}} = \sqrt{\frac{25}{4}} = \sqrt{6\cdot25} = 2\cdot5 =$ radius of the circle. $\frac{\sqrt{6\cdot25}}{2} = \sqrt{\frac{6\cdot25}{2^2}} = \sqrt{\frac{6\cdot25}{4}} = \sqrt{1\cdot5625} =$ semi-radius of the circle; therefore, the mean proportional between the diameter and semi-radius of the circle $= \sqrt{(\sqrt{25} \times \sqrt{1\cdot5625})} = \sqrt{(\sqrt{25 \times 1\cdot5625})} = \sqrt{(\sqrt{39\cdot0625})} = \sqrt{6\cdot25} = 2\cdot5 =$ radius of the circle.

Proof: $\sqrt{25} = 5 =$ diameter of the circle; and $\frac{\text{Diameter}}{2} = \frac{5}{2} = 2\cdot5 =$ radius of the circle.

Again:

Let F C, the diameter of the circle, be represented by the *mystic*[*] number 4.

Then:

$FC^2 = 4^2 = 16 =$ area of the square G B H K circumscribed about the circle. But, 4 times $4 = 16 =$ the perimeter of the circumscribing square G B H K; and it follows, that $\frac{\text{Area of the square G B H K}}{\text{Perimeter of the square G B H K}} = \frac{\text{Diameter of the circle}}{4}$; that is, $\frac{16}{16} = \frac{4}{4} = 1 =$ semi-radius of the circle; therefore, the mean proportional between the diameter and semi-radius of the circle $= \sqrt{4 \times 1} = \sqrt{4} = 2 =$ radius of the circle. But, in this ***particular*** and ***unique*** case, 1 is not only the arithmetical value of the semi-radius of the circle, but is also the arithmetical value of the area of a square on the semi-radius of the circle;* and it follows of necessity, that .16 times the area of a square on the semi-radius of every circle, is equal to the area of a square circumscribed about the circle. It is axiomatic, if not self-evident, that the area of a square on the radius of a circle is equal to 4 times the area of a square on the semi-radius; and because $\pi\ (r^2)$ = area in every circle, whatever be the value of π; it follows of necessity, that $4\pi (s\ r^2)$ = area in every circle, whatever be the value of π.

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[*] The critical reader may perhaps object to my use of the word *mystic*. My apology is this: the learned Professor De Morgan gave me the cue, by employing the term to the number 7, in his Budget of Paradoxes, No. 1. (See, *Athenæum*, Oct. 10, 1863). The peculiar properties of the number 4, however, have hitherto been obscured from the ken of Mathematicians, and for this reason, may fitly be termed *mystic*.

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* Mathematicians lose sight of the fact—or at any rate, they utterly ignore it, in their application of Mathematics to pure Geometry—that 1, $\sqrt{1}$, and 1^2, are definite arithmetical expressions, and are *arithmetically* of equal value. Mathematicians also ignore the following facts in their application of Mathematics to pure Geometry. First: If the diameter of a circle be represented by the *mystic* number 4, the circumference and area are *arithmetically* equal, whatever be the value of π. Second: If the circumference of a circle be represented by the *mystic* number 4, the diameter and area are *arithmetically* equal, whatever be the value of π. Third: In theoretical Geometry, a surface or superficies is said to have length and breadth, without thickness; and it is obvious, or self-evident, that a surface or superficies in theoretical Geometry must be defined by lines. Now, neither in *theoretical* nor *practical* Geometry, can two *straight* lines enclose a surface or superficies; but, two lines may enclose a superficies. For example: With any radius describe a circle, and draw a diameter. It is self-evident, that the diameter of the circle divides the area of the circle into two equal parts, and that the diameter is common to both. Hence: If the diameter of the circle be represented by the number 2, the ratio between the diameter of the circle and the area of that part of the circle on either side of it is, as 2 to $\frac{\pi}{2}$, whatever be the value of π. But, it is also self-evident, that the diameter of the circle divides the circumference into two equal parts; and it follows, that the ratio between the diameter and semi-circumference of a circle is as 2 to π, when the diameter of the circle = 2, whatever be the value of π. These facts are *unique*, and consequently, the ratios do not hold good of any circle, of which the diameter is either greater or less than 2.

Hence:

In the analogy or proportion, $A : x :: y : B$; if $A = 16 : x = \frac{4}{10} : y = 4\pi :$ and $B = \frac{\pi}{10}$; then, the product of the *means* is equal to the product of the *extremes;* and it follows, that the mean proportional between x and y, *the means*, is equal to the mean proportional between A and B, *the extremes;* that is, $\sqrt{x \times y} = \sqrt{A \times B}$, whatever *finite* and *determinate* arithmetical value, intermediate betwen 3 and 3·2, we may put upon π: and upon the shewing of Mathematicians, π must be greater than 3 and less than 3·2.

Now, I frankly admit that this proves nothing as to the true arithmetical value of π; but, it certainly proves that, whatever be the value of π, it cannot be an *indeterminate* arithmetical quantity, and therefore cannot be represented *arithmetically* by *any* infinite series.

If you, Reader, can prove my data to be inadmissible, or my reasoning to be illogical, and so, vitiate my conclusion, that π *cannot be* truly represented *arithmetically* by any *infinite series*, you need not trouble yourself to wade through the following pages. I shall assume, however, that you are *a sincere and earnest enquirer after scientific truth,* "*a reasoning geometrical investigator,*" *and an honest man.* If I am right in this assumption, you will read the following correspondence, and you will find that I have demonstrated in a great variety of ways, the truth of the THEORY, that 8 circumferences = 25 diameters in every circle, which makes $\frac{25}{8} = 3·125$, the true arithmetical value of π.

The algebraical formula, or in other words, the equation or identity, $4\left(\frac{\pi}{2}\right)^2 = \pi^2$, whatever be the value of π.

This fact may be demonstrated by means of any hypothetical value of π intermediate between 3 and 3·2, so that it be *finite* and *determinate.* For example: Let $\pi = 3{\cdot}1416$, a very close approximation to the arithmetical value assigned to it by "*recognised Mathematicians.*" Then: $4\left(\frac{\pi}{2}\right)^2 = 4\left(\frac{3{\cdot}1416}{2}\right)^2 = 4(1{\cdot}5708^2) = 4(2{\cdot}46741264)$ $= 3{\cdot}1416^2 = 9{\cdot}86965056$. Hence: 12½ times $\frac{\pi}{50} = \frac{\pi}{4}$, and this equation or identity = area of a circle of diameter unity, whatever be the value of π. How, then, is it possible, that the value of π can be an *indeterminate* arithmetical quantity? Let that *unscrupulous critic,* Professor de Morgan, go HONESTLY to work and prove that, "*James Smith, Esq., of Liverpool, is nailed by himself to the barn-door, as the delegate of miscalculated and disorganised failure.*" (See, *Athenæum,* July 25, 1868: Article: *Our Library Table.*)

You, Reader, may be an Arithmetician, without being either a Geometrician or a Mathematician. If so, you will be able to convince yourself, and, indeed, can hardly fail to perceive that, whether I am *right* or *wrong,* Mathematicians *must* be *wrong,* in assigning to π the *indeterminate* arithmetical value, 3·14159265, &c.; and I would recommend you to peruse my Letter of the 10th October, 1867, to the Editor of the *Athenæum.* (*See Appendix B,* page 286).

JAMES SMITH.

THE GEOMETRY OF THE CIRCLE
AND
MATHEMATICS
AS APPLIED TO GEOMETRY BY MATHEMATICIANS,
SHEWN TO BE
A MOCKERY, DELUSION, AND A SNARE.

To G. G. STOKES, ESQ., M.A., F.R.S., D.C.L.,
Lucasian Professor of Mathematics in the University of Cambridge.

SIR,

"*The British Association for the Advancement of Science*" has had its Royal, Aristocratical, Astronomical, Mechanical, Geological, Geographical, Botanical, and other philosophical Presidents, but it is many years since it had a professional "*recognised Mathematician*" for its President; and upon you that honour has fallen for the ensuing year.

That Euclid is inconsistent with himself, and Mathematics—as applied by Mathematicians—at fault, may be made as manifest to you, as that the three angles of a plane triangle are together equal to two right angles. In furnishing you with the proofs, I, as an old Life Member of the *British Association*, shall have done my duty; and it will only remain for you—as President, and guardian of its interests for a season—to do yours.

My pamphlet, "*Euclid at Fault*," (*see Appendix A*,) was fiercely attacked by numerous Mathematicians, and I know of no method more likely to convince you of Euclid's inconsistency, and that the Science of Mathematics, as applied by "*recognised Mathematicians*" to Geometry, is "*a mockery, delusion, and a snare*," than by giving you the correspondence which has passed between me and *one* of the many "*dragons*"* that has beset my "*upward path*," in the course of my scientific and literary career.

THE REV. PROFESSOR WHITWORTH *to* JAMES SMITH.

(Private.)

QUEEN'S COLLEGE, LIVERPOOL,
9th November, 1868.

SIR,

I suppose I am a "recognised Mathematician." As such, may I express a hope that you will take an early opportunity of withdrawing the accusation which you bring against "recognised Mathematicians," in your work "Euclid at Fault," as quoted in the *Liverpool Leader*, September 19th, 1868.

You say :—" Nothing short of proving the 47th Proposition of Euclid's First Book inconsistent with some other Proposition, would ever convince a recognised Mathematician that the arithmetical value of π is a finite and determinate quantity."

Sir, I think it is a shame that we should be thus maligned. My University degree, as well as my present office, entitle me to claim to be a recognised Mathematician ; and I *know*, and *always teach*, that the value of π is a finite and determinate quantity. If you have been so unfortunate as to meet with "recognised Mathematicians" who doubted of this truth, it is not just that you should impute the same ignorance to us all—or, at least, if you do, you should not *publish* the libel.

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* See the latter part of my Letter of the 7th November, 1866, to the Editor of the *Athenæum* (Appendix B).

Yours faithfully,

W. ALLEN WHITWORTH, M.A.,
Fellow of St. John's College, Cambridge; Professor of Mathematics in Queen's College, Liverpool.

JAMES SMITH *to* THE REV. PROFESSOR WHITWORTH.

BARKELEY HOUSE, SEAFORTH,
10*th November*, 1868.

SIR,

I am in receipt of your favour of yesterday, and, in reply, may observe :—I have long understood, from the teachings of "*recognised Mathematicians*," that the arithmetical value of π is 3·14159265, with a never-ending string of decimals, and therefore neither finite nor determinate.

You say :—" I *know*, and *always teach*, that the arithmetical value of π is a finite and determinate quantity."

My value of π is $\frac{25}{8} = 3·125$, which makes 8 circumferences exactly equal to 25 diameters in every circle ; and I can demonstrate that this is the true arithmetical value of π, by inductive and deductive reasoning, by constructive Geometry, by Logarithms, and by "*the rules of logic and common sense*," when applied to "*that indispensable instrument of science, Arithmetic.*"

Will you be kind enough to tell me, what is that "*arithmetical value of* π" by which you "*always teach*," and which you say "*is a finite and determinate quantity ?*"

I cannot recognise the right of "*private*" attack upon a *public document*, and therefore reserve to myself the right to make use of your Note in self-defence, if I find it necessary ; and the tone of your communication justifies this determination.

Yours faithfully,

JAMES SMITH.

The following quotations are from one of my Letters, in the correspondence referred to in the early part of my pamphlet "*Euclid at Fault.*" That correspondence is in print, but has not yet passed out of my publisher's hands. I shall wait the result of this communication to you, before I make it public:—

"Mr. R—— observes:—'*Now, that* X : Z : : 1 : $3\frac{1}{8}$, *is a result so simple, that you have only to state the proportion to see it. Circle area being as the squares of radii,* X : Z : : 16 : 50 : : 1 : $3\frac{1}{8}$. *π does not appear, need not appear.*' Does not $3\frac{1}{8}$ appear? And since $3\frac{1}{8}$ (X) = Z, by whatever finite value of π we get the values of X and Z (X *and* Z *denoting the areas of circles of which the radii are* 4 *and* 5 *times* $\sqrt{2}$), how can the value of π be indeterminate? Surely Mr. R—— should have exercised a little reflection, before he said of my conclusion, '*This is perfect nonsense.*'"

"Now, my dear Sir, mark the dilemma into which Mr. R—— has brought himself! He will not venture to dispute, that the area of a circumscribing square, to any circle, is found by dividing the area of the circle by $\frac{\pi}{4}$, whatever be the value of π. Now, if *a* denote the area of a circle, and *b* the area of a circumscribing square, then, $\frac{a}{\frac{1}{4}\pi} = b$, whatever be the value of π; and it follows, of necessity, that if π be indeterminate, *b* must also be indeterminate. The conclusion to which Mr. R——'s argument would lead us would be this: the diameter of an inscribed circle to any square cannot be arithmetically expressible, either by a finite number, or the square root of a finite number. Mr. R—— should have thought of this, before he ventured to say of my conclusion, '*This is perfect nonsense.*'"

"Well, then, Mr. R—— admits the following facts:—'*No other proof, but the simple statement of the ratios, is needed to show that* $3\frac{1}{8}$ (X) = Z, *when the radius of* X = 4, *and the radius of* Z = $5(\sqrt{2})$'—and he then puts the question:—'*But how does this upset the fact that π is indeterminate?*' Surely Mr. R—— will now see that in *assuming* it to be a *fact* that π is indeterminate, he might as well assume that the area of a square is indeterminate! Could absurdity go further?"

You, Sir, will not venture to tell me, that in a controversy with a "*recognised Mathematician*" on the ratio of diameter to circumference in a circle, the following is an unfair question to put to him:—Does the mathematical symbol π denote a determinate or an indeterminate arithmetical quantity? Well, then, with reference to this question, we have here two "*recognised Mathematicians*," one asserting that *the value of π is a finite and determinate quantity*, and the other maintaining it *to be a fact that π is indeterminate:* and so, differing from each other, even more widely than our *recognised Astronomers* differ among themselves, as to the Sun's distance from the Earth.

THE REV. PROFESSOR WHITWORTH *to* JAMES SMITH.

(Private.)

QUEEN'S COLLEGE, LIVERPOOL,
11*th November*, 1868.

SIR,

For your private information and personal guidance, I took the trouble of laying before you, in my former private Letter, the fact that you were incorrect, in ascribing to all recognised Mathematicians, the denial that π is a finite and determinate quantity. I supported my statement by telling you, that I myself both know and always teach that π is finite and determinate. I ventured to express a hope that you would take an opportunity of withdrawing a statement so defamatory of us Mathematicians, when you learnt that it rested on a mis-apprehension on your part.

I made no attack upon you whatever, and refrained from discussing anything but the one fact, that Mathematicians do not deny that π is finite and determinate. Therefore, I am rather surprised to receive your Letter of yesterday, in which you say that you cannot recognise the right of "private" attack upon a "public document," and therefore reserve to yourself the right to make use of my Note in self-defence, &c.

This sounds very like a *threat* to make my private communication public, or to treat it as public. But surely you do not argue that because a letter refers to a subject which has been made public, the writer thereof has no right to offer it as a "private" suggestion to the person to whom he addresses it. If you did not choose to receive a "private" communication from me, you might have declined to take any notice of my Letter, or might even have returned it. But even the morality of Liverpool would hardly sanction you in taking that which was offered as a private communication and treating it as public.

Truly I must be mis-interpreting your last paragraph. But a doubt upon this point must so obviously shake all confidence between us, that I must defer any other communication until it is cleared up.

Yours truly,
W. ALLEN WHITWORTH.

JAMES SMITH *to* THE REV. PROFESSOR WHITWORTH.

BARKELEY HOUSE, SEAFORTH,
12th November, 1868.

SIR,

Your favour of the 11th inst., under cover of an envelope bearing the Liverpool post-mark of to-day, only came into my hands this evening.

Had you favoured me with that "*finite and determinate arithmetical value of* π," which you "*know*" and by which you say you "*always teach*," there might, and, indeed, would have been, a reasonable excuse for your *ad miserecordiam* appeal.

It appears to me, that the *commercial morality* of Liverpool is exactly on a par with the *scientific morality* of "*recognised Mathematicians*," and the sooner both are exterminated, the better for Society and for Science.

As matters stand, it is my intention to post a letter on Saturday, to one of your (St. John's) College *chums*, in which I shall give your communications, and my answers, *in extenso*.

Yours faithfully,
JAMES SMITH.

THE REV. PROFESSOR WHITWORTH *to* JAMES SMITH.

16, PERCY STREET, LIVERPOOL,
13th November, 1868.

SIR,

You totally misinterpret my Letter when you speak of an "*ad miserecordiam* appeal." If I appealed at all it was rather *ad honorem.*

Of course any breach on your part of that confidence under which I addressed you, would not in its direct effects be of the slightest consequence to me. I hold no secret opinions. But it is of the greatest consequence to me that I should not unwittingly enter into correspondence with anyone who would violate the sanctity of a confidential communication. Therefore, in order to test my position, I must decline for the present, to sanction your wish to lay my "private" Letters before the third person to whom you refer in such extraordinary terms.

If you wish our correspondence to continue further, will you please to assure me that such Letters as I may mark "private" will be respected as such—unless my permission is gained to treat them otherwise. On such an understanding only do I correspond with anyone, but I have never before been obliged to give expression to such a stipulation.

I am, Sir,
Yours faithfully,
W. ALLEN WHITWORTH.

JAMES SMITH *to* THE REV. PROFESSOR WHITWORTH.

BARKELEY HOUSE, SEAFORTH,
14th November, 1868.

SIR,

I was born in Liverpool, and have spent all my days in Liverpool and its immediate neighbourhood, and you *know* what "*Liverpool morality*" is: why then, should you expect an appeal "*ad honorem*" to such an one as I am, to be of any effect? Bad as I am, however, it is a consolation to me to *know* that nothing I can say or do, can, "*in its direct effects, be of the slightest consequence to you.*"

Your original communication, judged "*by the rules of logic and common sense,*" was not written with the intention of kindly tendering information and advice. Could I have so interpreted it, our correspondence would have taken a very different turn. Had you known me better, you would have known, that I may be led, but cannot be driven.

If you "*know*" and "*always teach*" by the "*finite and determinate*" value of $\pi = \frac{25}{8} = 3·125$, you are a proselyte to my discovery: if not, what mean your assertions? For, if you teach by the "*finite and determinate*" value of $\pi = 3·1416$, or 3·14159, or 3·14159265, or by π = the integer 3 with any other string of decimals, are not your assertions mere quibbles?

If you have "*unwittingly*" entered into correspondence with *one* of the "*Liverpool morality*" school, the fault is yours not mine: there is no occasion for you to continue it, and you may rest assured that, if you leave me alone, I shall not trouble you.

In due time the scientific public will learn through the press, that I am not the only Mathematician that *knows*, and *always teaches*, by a *finite and determinate value of* π; and this will afford you the opportunity of attacking me publicly, if I "*publish a libel.*"

I am, Sir,
Yours faithfully,
JAMES SMITH.

I certainly did not expect to hear again from the Reverend and learned Professor, and at the time was very unwilling to be drawn into a correspondence with him. He chose, however, not to let the controversy rest here, and the following was his next communication:—

9

THE REV. PROFESSOR WHITWORTH *to* JAMES SMITH.

LIVERPOOL, *November 16th*, 1868.

SIR,

I am amused at your first paragraph (of November 14th), in which you remonstrate with me, for attributing to you the quality of honour.

Your Letter contains many misrepresentations, which I pass over, as I am weary of the contentious tone which you persist in giving to our correspondence. My object is not contention, my only desire being that truth should prevail. Any argument must cease to be edifying, as soon as either party seeks for victory, rather than for the TRUTH.

Twice in your Letter of the 14th November, you speak of me as teaching by a finite value of π: and on looking back upon your other Letters I observe, that whenever you have occasion to quote my original statement, you uniformly garble it by the insertion of the preposition BY. I do not teach *by a* value of π at all,* but I *know* and *always teach*, that the value of π is neither infinite nor indefinite.

I never attacked you as to the accuracy or inaccuracy of the value you assign to π. I merely protested (as I do still) against the false statement that recognised mathematicians assert π to be either infinite or indefinite, or that the value which they assign to it *is* either infinite or indefinite. The ratio of two finite lengths cannot be infinite, and a quantity given by a definite convergent series such as, $\pi = \frac{16}{\sqrt{3}}\left(\frac{1}{1\cdot3} + \frac{2}{5\cdot7}\cdot\frac{1}{9} + \frac{3}{9\cdot11}\left(\frac{1}{9}\right)^2 + \frac{4}{13\cdot15}\left(\frac{1}{9}\right)^3 + \text{\&c.}\right.$, cannot be indefinite.

You seem determined to thrust upon me your notion that $\pi = 3\cdot125$. I did not propose to discuss this, but if ever I have the pleasure of a personal interview with you, I will prove, in five minutes, that your value is incorrect; and in fifteen, that our usually assigned value is correct.

---

* If it be true that Professor Whitworth does not teach "*by a value of π at all*," how does he teach his pupils to find the circular measure of an angle? Is not the circular measure of an angle of great importance in the theory of Mathematics?

As I do not possess any of your published works, I do not know how you have been led astray. But I should like to enquire whether you have ever discovered a flaw in the reasoning by which it is proved that π has the finite and definite value which we Mathematicians assign to it.

Believe me, dear Sir,
Faithfully yours,
W. ALLEN WHITWORTH.

JAMES SMITH *to* THE REV. PROFESSOR WHITWORTH.

BARKELEY HOUSE, SEAFORTH.
17*th November*, 1868.

DEAR SIR,

Your favour of yesterday is courteous in its tone, which is more than could be said of your first communication, in which you *branded* me as a *maligner* and *libeller.*

You now say:—"*My object is not contention, my only desire being that truth should prevail. Any argument must cease to be edifying as soon as either party seeks for victory, rather than for the TRUTH.*" Granted! But, my Dear Sir, the "*contention*" has been on your part, not mine: and you could not have advanced a stronger argument to show that our correspondence had better cease. My only desire is "*that truth should prevail;*" and—"*Magna est veritas,*" and prevail it will.

With reference to the third paragraph in your favour of yesterday, I have only to say, the following is a literal quotation from your first communication. "*I know*, and *always teach*, that the arithmetical value of π is a finite and determinate quantity." *

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* Is not the meaning of the words *finite and determinate*, the opposite of that attached to the words *infinite and indefinite?*

With reference to the fourth paragraph, I may observe, that in putting the question:—"What is that value of π, which you know, by which you always teach, and which you say is finite and determinate?—I was necessitated by the rules of grammar, to make use of the preposition BY: and in doing so, I certainly did not garble your original statement."*

In the fifth paragraph you say:—"*The ratio of two finite lengths cannot be infinite*, (True!) *and a quantity given by a definite convergent series, such as*, $\pi = \sqrt{\frac{16}{3}}\left(\frac{1}{1\cdot 3} + \frac{2}{5\cdot 7}\cdot\frac{1}{9} + \frac{3}{9\cdot 11}\cdot\left(\frac{1}{9}\right)^2 + \frac{4}{13\cdot 15}\left(\frac{1}{9}\right)^3 + \&c.\right.$, *cannot be indefinite.*" This I cannot understand. How, "*by the rules of logic and common sense*," can a convergent series, or in other words, how can a never ending series, which is indefinite, be arithmetically determinate?

With reference to the sixth paragraph, I have only to say that to me it is *amusing*.

With regard to the first part of the concluding paragraph, I may observe, that you would have done well to make yourself master of how it happened, that I was led astray—if, indeed, it be a fact that I have been led astray—on the ratio of diameter to circumference in a circle: and with regard to the latter part of it, I may tell you, that I can point out the "*flaw in the reasoning*," by which Orthodox Mathematicians fancy they prove, "*that π has the finite and definite value which they assign to it.*"

I have corresponded since the year 1860, with four "*recognised Mathematicians*," for periods varying from six to eighteen months; and think I am thoroughly acquainted with every argument that can be advanced from the Orthodox point of view, against the truth of the THEORY, that 8 circumferences = 25 diameters in every circle. I am preparing a work for the press, (as you know,) which with my public duties keeps me fully employed, and I am unwilling to have another long correspondence, in which I should only have to travel over much of the same ground again.

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* It appears to me, I could only be charged with garbling the learned Professor's statement, on the supposition, that π is a thing of no value when found, and that no use can be made of it to teach anything in Mathematics. Is not Trigonometry a branch of Mathematics?

I should be happy to make your personal acquaintance, and if you will come out on Saturday next and take a friendly dinner with us, at 6-30, I shall be glad to see you, and I think I can shew you something that will astonish you.

Believe me, my dear Sir,
Very truly yours,
JAMES SMITH.

The morning's post, of the 17th November, brought me the following communication, to which the concluding paragraph of the last Letter refers:—

RUGBY, *November 16th*, 1868.

MY DEAR SIR,

I have received your two somewhat extended Letters, and beg to acknowledge them. I cannot say that I have perused them with an attention at all proportionate to their length; but I have been able to perceive that you are able to prove your conclusions by a process of reasoning which is absolutely sound and logical, and that therefore your conclusions are as certain as their premises, with which they are in fact identical.

This will, I hope, be accepted as my recantation, and be published along with your Letters to me.

Sir, I admire your indomitable perseverance, your hand-writing, and your liberal expenditnre of postage stamps. They are worthy of a more extended success than they have yet met with. But pray accept, as an instalment of the debt that will not be fully paid in your life-time, my admission above made. You are obtaining recruits at last from the ranks of professed Mathematicians.*

Believe me, respectfully yours,
JAMES M. WILSON.

* There can be no doubt, that Mr. James M. Wilson professes to be a "*gentleman*" and "*a man of honour.*" *(See Appendix C.)*

Mr. Whitworth would not drop the controversy, and thus I was most unwillingly drawn into a correspondence with him.

"*By the rules of logic and common sense,*" it is obvious or self-evident, that in every circle there must be an arc equal to radius. For, if not self-evident, it is axiomatic, that in every circle, 6 times radius = the perimeter of an inscribed regular hexagon. But, it is self-evident, that the circumference of a circle is greater than the perimeter of its inscribed regular hexagon, and it follows of necessity, that an arc must be greater than its chord.

Now, $\frac{24}{25}(360) = 6(57{\cdot}6)$, that is, $\frac{24 \times 360}{25} = 6(57{\cdot}6)$ or, $\frac{8640}{25} = 345{\cdot}6$: and on the THEORY that 8 circumferences = 25 diameters in every circle, 345·6 is the perimeter of an inscribed regular hexagon, and $\frac{345{\cdot}6}{6} = 57{\cdot}6$, the value of an arc equal to radius, when the circumference of a circle = 360. Now, the perimeter of a regular inscribed hexagon to a circle of radius 1 = 6, and, by analogy or proportion, $345{\cdot}6 : 360 :: 6 : 6{\cdot}25$. But, $8(6{\cdot}25) = 25$ (diameter); that is, $(8 \times 6{\cdot}25) = (25 \times 2) = 50$; and it follows of necessity, that 6·25 is the circumference of a circle of radius 1, on the THEORY that 8 circumferences = 25 diameters in every circle.

The area of a circle of radius 1, and the semi-circumference of a circle of radius 1, are represented by the same arithmetical symbols, whatever be the value of π: and the area of a circle of radius 1, and the circumference of a circle of diameter unity, are represented by the same arithmetical symbols, whatever be the value of π: and it follows of necessity, that $\frac{6{\cdot}25}{2} = 3{\cdot}125$, is the circumference of a circle of diameter unity; on the THEORY that 8 circumferences = 25 diameters in every circle.

Now, $\frac{24}{25}\,(3{\cdot}125) = 6\,({\cdot}5)$, that is, $\frac{24 \times 3{\cdot}125}{25} = 6\,({\cdot}5)$, or, $\frac{75}{25} = (6 \times {\cdot}5) = 3 =$ the perimeter of a regular inscribed hexagon to a circle of diameter unity, whatever be the value of π. But, $\frac{25}{24}\,({\cdot}5) = \frac{25 \times {\cdot}5}{24} = \frac{12{\cdot}5}{24} = {\cdot}52083333$, with 3 to infinity: and, $\left({\cdot}52083333\ldots - \frac{{\cdot}52083333\ldots}{25}\right) = ({\cdot}52083333\ldots - {\cdot}02083333\ldots) = {\cdot}5 =$ radius. But, $6\,({\cdot}52083333\ldots) = 3{\cdot}12499998\ldots$, and we may make the approximation to $3{\cdot}125$ as close as we please, or, at any rate within 2 at the last place of decimals, to whatever extent we may carry the number of decimals. Hence: by analogy or proportion, $3 : 3{\cdot}125 :: 345{\cdot}6 : 360$, and we arrive at a similar result whether we make 360 or $3{\cdot}125$ our "*initial position*" or starting point; on the THEORY that 8 circumferences = 25 diameters in every circle.

Well, then, having ascertained these facts, we can now prove a fallacy in the method by which "*recognised Mathematicians*" fancy they arrive at the ratio of diameter to circumference in a circle.

By hypothesis, let the circumference of a circle of diameter unity $= 3{\cdot}1416$, and this is a close approximation to its true arithmetical value, according to "*recognised Mathematicians.*" Then: $\frac{24}{25}\,(3{\cdot}1416) = 6\,({\cdot}502656)$, that is, $\frac{24 \times 3{\cdot}1416}{25} = 6\,({\cdot}502656)$ or, $\frac{75{\cdot}3984}{25} = 3{\cdot}015936$. But, $3{\cdot}015936$ is greater than the known and indisputable arithmetical value of the perimeter of a regular inscribed hexagon to a circle of diameter unity. Now, (circumference × semi-radius) = area in every circle; therefore, 3·1416 × ·251328 = ·7895720448; and ·7895720448 is the area of a circle of which the circumference is 3·1416, on the hypothesis that π (which denotes the circumference of a circle of diameter unity) = 3·1416; and is greater than $\frac{\pi}{4}$; that is, greater than $\frac{3·1416}{4} = ·7854$. But, $\frac{\pi}{4}$ = area of a circle of diameter unity, whatever be the value of π. Hence: $3\frac{1}{8}$ ($·502656^2$) = (3·125 × ·252663054336) = ·7895720448, and it follows of necessity, that ·7895720448 is the true area of a circle of which the radius = ·502656: and $\frac{3·125}{4} = ·78125$, the true area of a circle of diameter unity. Well, then, does not this make Mathematics—as applied by "*recognised Mathematicians*"—a delusion and a snare? With these facts present to your mind, what follows will be as plain and intelligible to you, as that (2 + 2) = 4.

THE REV. PROFESSOR WHITWORTH *to* JAMES SMITH.

LIVERPOOL, *Nov.* 18*th*, 1868.

DEAR SIR,

As I have to officiate in church on Saturday night, as well as to prepare two sermons for Sunday, I am unable to accept your polite invitation.

I am glad to see that you yield one half of my original assertion, in granting, that π (as represented by Mathematicians) is *finite.*

You will as readily grant the other half, that it is *definite* if you consider the meaning of this term. A magnitude is *definite* when there is no doubt as to its value, so that we are able to determine whether it is greater or less than any assigned magnitude. Thus $\sqrt{2}$

is a *definite* quantity. We can say that it is less than $\frac{10}{7}$, and yet greater than $\frac{7}{8}$; and so we can always answer the question :—Is it greater or less than x, whatever x may be ?

You ask how can a never ending series which is indefinite be arithmetically determinate. I answer that it can, just as the never ending series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} +$ &c., represents to us the determinate arithmetical value 2.

But of course this involves the definition of convergency and the definition of the term "sum to infinity." When a series is such that by sufficiently increasing the number of terms the sum tends towards a finite quantity, and can be made (by sufficiently increasing the number of terms) to differ from that finite quantity, by less than *any assignable* quantity, then the series is said to be convergent, and the finite quantity is called the limit or the sum to infinity of the series.

As an Arithmetician, do you deny that the recurring decimal $\dot{1}4285\dot{7}$ represents the vular fraction $\frac{1}{7}$? If it does, we have the determinate value $\frac{1}{7}$ for the never ending series $\frac{142857}{1000000} + \frac{142857}{1000000000000}$ + &c., *ad infinitum*.

Any way, if the received value of π is wrong, the trigonometrical expansion of tan $'x$ is wrong. The whole theory of series is upset, the results of the lunar and planetary theory are utterly incorrect; and it can only be by accident that eclipses, &c., have recently occurred at the predicted times. I think perhaps one of the most convincing tests which we could well apply to the question in which you are interested, would be furnished, if you would calculate the time of the next lunar eclipse according to your value of π. *We* expect it on January 28th, 1869, but if your value of π is correct we are quite wrong.

Will you tell me in which of your publications I can find a reference to the flaw which you speak of having discovered in the method by which it is proved that $\pi = \frac{16}{\sqrt{3}} \left(\frac{1}{1 \cdot 3} + \frac{2}{5 \cdot 7} \cdot \frac{1}{9} + \frac{3}{9 \cdot 11} \left(\frac{1}{9}\right)^2 + \right.$ &c.

Believe me, dear Sir,

Very truly yours,

W. ALLEN WHITWORTH.

JAMES SMITH *to* THE REV. PROFESSOR WHITWORTH.

BARKELEY HOUSE, SEAFORTH,
23rd November, 1868.

DEAR SIR,

I am sorry I named an inconvenient day, in asking you to take a friendly dinner with me ; but if you will fix your own day, and drop me a note the day before, I shall be most happy to see you, and make your personal acquaintance. My engagements have prevented me replying sooner to your favour of the 18th inst.

You say :—"*I am glad to see that you yield one-half of my original assertion, in granting that π (as represented by Mathematicians) is finite.*" In this observation you "*misinterpret*" me.

You next say :—"*You will as readily grant the other half, that it is definite, if you consider the meaning of the term. A magnitude is definite when there is no doubt as to its value, so that we are able to determine whether it is greater or less than any assigned magnitude.*" You illustrate your meaning by observing—"*Thus,* $\sqrt{2}$ *is a definite quantity. We can say that it is less than* $\frac{10}{7}$, *and yet greater than* $\frac{7}{5}$; *and so, we can always answer the question—Is it greater or less than x, whatever x may be?*"

Now, I *know* that $\sqrt{2}$ is a definite arithmetical expression, and that it represents the value of the diagonal of a square, of which the value of the sides is represented by the arithmetical symbol 1 ; but, I also *know* that, if we extract the root of the expression $\sqrt{2}$, to whatever extent we may carry the number of decimals in working out the calculation, there would still be a remainder, so that we cannot give a true and definite arithmetical expression—*decimally*—to $\sqrt{2}$.

Again : I *know*, that $\frac{\sqrt{2}}{10} = \frac{\sqrt{\cdot 5}}{5}$, and that this equation = ·1414213, &c. : but I also *know* that ·1414213, &c. is the arithmetical value of the natural sine of an angle of 8° 8′ : and I *know* that the sides that contain the right angle, in a right-angled triangle of which the acute angle is an angle of 8° 8′, are in the ratio of 7 to 1. You will find that in tables to 7 places of decimals, the sine of an angle of 8° 8′ is given as ·1414772; and I maintain, that this is not the true arithmetical value of the natural sine of an angle of 8° 8′.

You next observe:—" *You ask:—How can a never-ending series which is indefinite, be arithmetically determinate? I answer that it can, just as the never-ending series* $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} +$ &c., *represents to us the determinate arithmetical value 2.*"

Now, I *know*, that the arithmetical series $\frac{1}{17} + \frac{2}{17} + \frac{3}{17} + \frac{4}{17} + \frac{5}{17} + \frac{6}{17} + \frac{7}{17} + \frac{8}{17} + \frac{9}{17} + \frac{10}{17} + \frac{11}{17} + \frac{12}{17} + \frac{13}{17} + \frac{14}{17} + \frac{15}{17} + \frac{16}{17} + \frac{17}{17} = 1{\cdot}9999999999999992$., and is less than 2 by a distinctly assignable quantity, represented by the digit 8 at the sixteenth place of decimals. This series put in the form of an arithmetical sum stands as in the margin. You will observe that $\frac{1}{17}$, $\frac{2}{17}$, $\frac{3}{17}$, &c., are recurring decimals of 16 figures.

No doubt this arithmetical series "*represents to us the determinate arithmetical value 2.*" But, if we double the number of terms the result is very near a finite quantity, but not quite so near integral finity as in this series.

Permit me to direct your attention to the fact, that in this arithmetical sum, each row of figures, (omitting the last) whether taken vertically or horizontally, are composed of the same figures, but in different positions. Hence, any row of figures in this sum, whether taken vertically or horizontally, adds up to the same number = 8 times 9 = 72.

$\frac{1}{17} = {\cdot}0588235294117647$
$\frac{2}{17} = {\cdot}1176470588235294$
$\frac{3}{17} = {\cdot}1764705882352941$
$\frac{4}{17} = {\cdot}2352941176470588$
$\frac{5}{17} = {\cdot}2941176470588235$
$\frac{6}{17} = {\cdot}3529411764705882$
$\frac{7}{17} = {\cdot}4117647058823529$
$\frac{8}{17} = {\cdot}4705882352941176$
$\frac{9}{17} = {\cdot}5294117647058823$
$\frac{10}{17} = {\cdot}5882352941176470$
$\frac{11}{17} = {\cdot}6470588235294117$
$\frac{12}{17} = {\cdot}7058823529411764$
$\frac{13}{17} = {\cdot}7647058823529411$
$\frac{14}{17} = {\cdot}8235294117647058$
$\frac{15}{17} = {\cdot}8823529411764705$
$\frac{16}{17} = {\cdot}9411764705882352$
$\frac{17}{17} = 1{\cdot}0000000000000000$

$1{\cdot}9999999999999992$

You next put a question to me:—*As an Arithmetician, do you deny that the recurring decimal* ·142857 *represents the fraction* $\frac{1}{7}$?" My answer is, Certainly not! You then say:—"*If it does, we have the determinate value* $\frac{1}{7}$ *for the never-ending series* $\frac{142857}{1000000} + \frac{142857}{1000000000000}$ + &c., *ad infinitum.*"

Now, the series $\frac{1}{7} + \frac{2}{7} + \frac{3}{7} + \frac{4}{7} + \frac{5}{7} + \frac{6}{7} + \frac{7}{7}$ = 3·999997, and is a close approximation to the MYSTIC number 4. This series, put in the form of an arithmetical sum, stands as in the margin. Thus, upon your own shewing, this series must be taken as representing the determinate arithmetical value 4. You will observe that the figures in any row are the same as the figures in any other row, (omitting the last) whether taken vertically or horizonally.* They are simply changed in position; and separately, add up to 3 times 9 = 27: and 27 + 72 = 99, and is less that 100 times unity by 1.

$\frac{1}{7}$ = ·142857
$\frac{2}{7}$ = ·285714
$\frac{3}{7}$ = ·428571
$\frac{4}{7}$ = ·571428
$\frac{5}{7}$ = ·714285
$\frac{6}{7}$ = ·857142
$\frac{7}{7}$ = 1·000000
3·999997

Now, if we double the number of terms, that is to say, if we take the arithmetical series, $\frac{1}{7} + \frac{2}{7} + \frac{3}{7} + \frac{4}{7} + \frac{5}{7} + \frac{6}{7} + \frac{7}{7} + \frac{8}{7} + \frac{9}{7} + \frac{10}{7} + \frac{11}{7} + \frac{12}{7} + \frac{13}{7} + \frac{14}{7}$, the calculations worked out, give as the result, 14·999994, very nearly the finite arithmetical quantity 15, and no doubt this series must be taken as representing the integral arithmetical value 15. If we treble the number of terms, we get as the result 32·999921, very nearly the finite arithmetical quantity 33, and no doubt the arithmetical series of 21 terms from $\frac{1}{7} + \frac{2}{7}$ + &c., must be taken as representing the integral arithmetical value 33. We might go on with series of this description, *ad infinitum*, but at every step we should be receding from the finite quantity represented by the result of the original series. Well, then, I have given you two series, which by no extension of the number of terms can be made to tend towards an integral finite quantity, and differ from that finite quantity, by less than an assignable quantity.

You say:—"*Any way, if the received value of π is wrong, the trigonometrical expansion of tangent* ·x *is wrong, the whole theory of series is upset. The result of the lunar and planetary theory are utterly incorrect.*" In all this I quite agree with you. The received value of π is wrong. Hence it is, that Astronomers cannot agree among themselves as to the Sun's distance! How can they find the

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* If we were to double, treble, or quadruple the number of decimals representing the terms, it is self-evident, that the result of the sum would be, the integer 3, with the repeating decimal 9 to the last place of decimals, and less than 4 by an assignable quantity.

Sun's distance while they remain ignorant of the true value of π; or in other words, while they remain ignorant of the true ratio between the diameter and circumference of a circle? How can they find the Sun's parallax, until they know the true diameter of the earth; a thing essential to the discovery of the Sun's parallax?

You next say: "*It can only be by accident that eclipses, &c., have recently occurred at the predicted times.*" In this you are wrong. π does not enter into the calculations by which eclipses are foretold. Eclipses were predicted by the Chaldeans more than 3000 years ago. The eclipse you refer to will occur again, after 223 lunations, or after a period of about 18 years and 11 days. The period of 18 years and 11 days for 223 lunations was known to the Chaldeans. They only predicted an eclipse to the day, but the records of careful *observers* in more recent times, enable Astronomers *now* to predict an eclipse to a second of time.

Now, my dear Sir, you *know* that "Logarithms may be considered the indices or arithmetical series of numbers, adapted to the terms of a geometrical series, in such sort that 0 corresponds to 1, or is the index of it in the geometricals."

$$\text{"Thus}\begin{cases}0,\ 1,\ 2,\ 3,\ 4,\ 5,\ \text{\&c., indices or logarithms.}\\ 1,\ 2,\ 4,\ 8,\ 16,\ 32,\ \text{\&c., geometric progression."}\end{cases}$$

Hence:

The Logarithm of 2 is ·30102995 to the base 10, and is given in tables calculated to 7 places of decimals, as ·3010300. The Logarithm of 3 is ·477121255, and is given in tables as ·4771213. And these values of the Logarithms are sufficiently accurate for all practical purposes.

Well, then, in the arithmetical series, $\frac{1}{17} + \frac{2}{17} + \frac{3}{17}$ &c., to 17 terms, we obtain the following result: or in other words, worked out arithmetically we get as the result, 1·9999999999999992. Now, the Logarithm corresponding to the natural number 1·9999999999999992 is 0·3010300333333331656, calculated by the difference as given in Hutton's Tables, and this is the Logarithm of the *natural* number 2, given in tables as ·3010300, which is sufficiently accurate for all practical purposes.

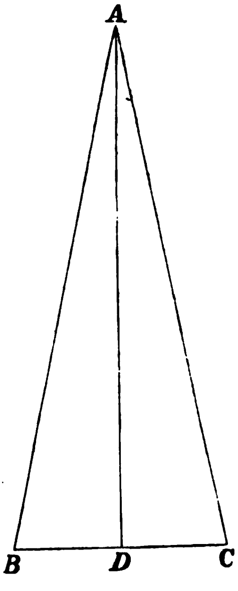
Now, if the circumference of a circle be represented by the arithmetical symbol 4, the diameter and area of the circle are represented by the same arithmetical symbols, whatever be the value of π. Again: in the analogy or proportion, A : B : : B : C, when A denotes $\frac{\pi}{4}$ and B denotes 1, then, C denotes the diameter and area of a circle of circumference = 4, whatever be the value of π. Again: π must be greater than 3 and less than 3·2, and on the THEORY that 8 circumferences of a circle are exactly equal to 25 diameters, $\frac{25}{8} = 3{\cdot}125$, is the arithmetical value of π. Well, then, I can demonstrate that this is the true value of π, by inductive and deductive reasoning; by constructive Geometry; by Logarithms; and by the "*rules of logic and common sense,*" when applied to "*that indispensable instrument of science, Arithmetic.*"

Well, then, in the analogy or proportion, A : B : : B : C, let A denote $\frac{3{\cdot}125}{4}$ and B denote 1. Then: ·78125 : 1 : : 1 : 1·28, and 1·28 is the value of the diameter and area in a circle of circumference = 4. But, 3 times diameter = perimeter of a regular inscribed hexagon to every circle, and $\frac{3}{\pi}$ expresses the ratio between the perimeter of every regular hexagon and the circumference of its circumscribing circle, whatever be the value of π; therefore, 3 (1·28) = 3·84 = perimeter of a regular inscribed hexagon to a circle of circumference = 4; and, 3 : 3·125 : : 3·84 : 4. But, 25 times 1·28 = 32; and 32 is the value of the area of an inscribed square to a circle of diameter 8; and the area of a circumscribing square is the double of the area of an inscribed square, in every circle; and the area of a circle is found by multiplying the area of a circumscribing square by $\frac{\pi}{4}$, whatever be the value of π. Now, if the diameter of a circle = 8, radius = 4, area of circumscribing square = 64, and area of inscribed square = 32. Hence: $3\frac{1}{8}(4^2) = \left\{\left(32 + \frac{32}{4}\right) + \frac{1}{4}\left(32 + \frac{32}{4}\right)\right\}$; that is, $(3{\cdot}125 \times 16) = \left\{40 + \frac{1}{4}(40)\right\}$; and this equation = area of the circle. But, $\left\{40 + \frac{1}{4}(40)\right\} = \left(64 \times \frac{\pi}{4}\right)$; that is, $\left\{40 + \frac{1}{4}(40)\right\} = (64 \times {\cdot}78125)$, and this equation = 50 = area of of the circle; and since we cannot get these equations by any other value of π, it follows that 3·125 must be the true arithmetical value of π, and makes 8 circumferences = 25 diameters in every circle.

I shall send you one of my Pamphlets along with this communication,* and I have a Letter, recently received from a "recognised Mathematician," in which he observes:—"*I can't understand why Mathematicians have not thoroughly examined your reasonings long before this. The demonstration on pp.* 14 *and* 15 *ought to have sufficed.*" You will find that, in this demonstration, I have established the *true* value of π, by hypothetically assuming a *false* value of it.

Let A B C denote one of 25 equal isosceles triangles inscribed in a circle, with the angle B A C and its opposite side bisected by the line A D. By hypothesis, let the circumference of the circle = 360.

Then: $\frac{24}{25}(360) = 6\,(57{\cdot}6)$, and this equation = 345·6, and is equal to the perimeter of a regular inscribed hexagon to a circle of circumference = 360; and by analogy or proportion, 345·6 : 360 : : 3 : 3·125. Now, $\frac{360}{25} = 14{\cdot}4 = 14^\circ\,24'$ = the angle B A C; therefore, $\frac{14^\circ\,24'}{2} = 7^\circ\,12'$ is the value of the angles D A B and D A C; and, $90^\circ - 7^\circ\,12' = 82^\circ\,48'$ is the value of the angles A B D and A C D, when the values of these angles are expressed in degrees; for, the angle B A C and its opposite side B C are bisected by the line A D, and it follows, that A D B and A D C are similar right-angled triangles.



Well, then, by hypothesis, A B C represents one of 25 equal isosceles triangles, inscribed in a circle of circumference = 360. Hence: $\frac{360}{25} = 14{\cdot}4 = \text{B C}: \frac{\text{B C}}{2} = \frac{14{\cdot}4}{2} = 7{\cdot}2 = \text{D B}$. $8\,(\text{D B}) = (8 \times 7{\cdot}2) = 57{\cdot}6 = \text{AB}$; therefore, $\text{A B}^2 - \text{D B}^2 = 57{\cdot}6^2 - 7{\cdot}2^2 = 3317{\cdot}76 - 51{\cdot}84 = 3265{\cdot}92 = \text{A D}^2$; therefore, $\sqrt{3265{\cdot}92} = 57{\cdot}148228\ldots = \text{A D}$: and the triangles A D B and A D C are *incommen-*

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* "The British Association in Jeopardy, &c."

surable right-angled triangles. But, $\frac{DB}{AB} = \frac{7\cdot2}{57\cdot6} = \cdot125$, and ·125 is the natural sine of the angles D A B and D A C. $\frac{AD}{AB} = \frac{57\cdot148228}{57\cdot6} = \cdot9921567$, and ·9921567 is the natural sine of the angle A B D; and *you* will admit that the sines and cosines of angles are the complements of each other. Now, the Logarithm corresponding to the natural number ·125 is 9·0969100; and this is the Log.-sin. of the angle D A B. The Logarithm corresponding to the natural number ·9921567 is 9·9965802; and this is the Log.-sin. of the angle A B D.*

Let the length of A B, the side subtending the right angle in the right-angled triangle A D B, be represented by any finite arithmetical quantity, say 77, and be given to find the lengths of the other two sides D B and A D, and prove that the sides A B and D B are in the ratio of 8 to 1.

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* In this paragraph I have fallen into more than one lapsus. To clearly express my meaning it should have run as follows:—

Well, then, by hypothesis, A B C represents one of 25 equal isosceles triangles, inscribed in a circle of circumference = 360. Hence: $\frac{360}{25}$ = 14·4 = the angle B A C, $\frac{14\cdot4}{2} = 7\cdot2$ = the sine of angle D A B, to a circle of circumference = 360 = D B. 8 (D B) = 8 × 7·2 = 57·6 is the value of an arc equal to radius, to a circle of circumference = 360 = A B; therefore, $AB^2 - DB^2 = 57\cdot6^2 - 7\cdot2^2 = 3317\cdot76 - 51\cdot84 = 3265\cdot92 = AD^2$; therefore, $\sqrt{3265\cdot92} = 57\cdot148228\ldots = AD$, and the triangle A D B is an incommensurable right-angled triangle. But, $\frac{DB}{AB} = \frac{7\cdot2}{57\cdot6} = \cdot125$: and ·125 is the natural sine of the angle D A B. $\frac{AD}{AB} = \frac{57\cdot148228}{57\cdot6} = \cdot9921567$: and ·9921567 is the natural sine of the angle A B D; and you will admit that the sines and co-sines of angles are the complements of each other. Now, the Logarithm corresponding to the natural number ·125 is 9·0969100; and this is the Log.-sin. of the angle D A B. The Logarithm corresponding to the natural number ·9921567 is 9·9965802; and this is the Log.-sin. of the angle A B D.

The reader will observe that, in this Letter, I have employed the term "*natural sine*" in the same sense that is attached to it by Mathematicians; that is to say, that the *natural sine* and *trigonometrical sine* of an angle are *arithmetically* the same; but, in my Letter to his Grace the Duke of Buccleuch, I have proved that the natural, geometrical, and trigonometrical sines of an angle, may be—*arithmetically*—all different.

I shall *assume* that you admit D A B and A B D to be angles of 7° 12′ and 82° 48′; for, to dispute these facts would simply be equivalent to *asserting*, that $\frac{360}{25}$ is not equal to 14·4; or, that (2 + 2) is not equal to 4.

Then:

As Sin of angle D = Sin 90°	Log.	10·0000000
: the given side A B = 77.............................	Log.	1·8864907
: : Sin of angle D A B = Sin 7° 12′..................	Log.	9·9069100
		10·9834007
		10·0000000
: the required side D B = 77 × ·125 = 9·625..	Log.	0.9834007

Again:

As Sin of Angle D = Sin 90°	Log.	10·0000000
: the given side A B = 77..................................	Log.	1·8864907
: : Sin of angle A B D = Sin 82° 48′	Log.	9·9965802
		11·8830709
		10·0000000
: the required side A D = 77 × ·9921567 = 76·3960659	Log.	1·8830709

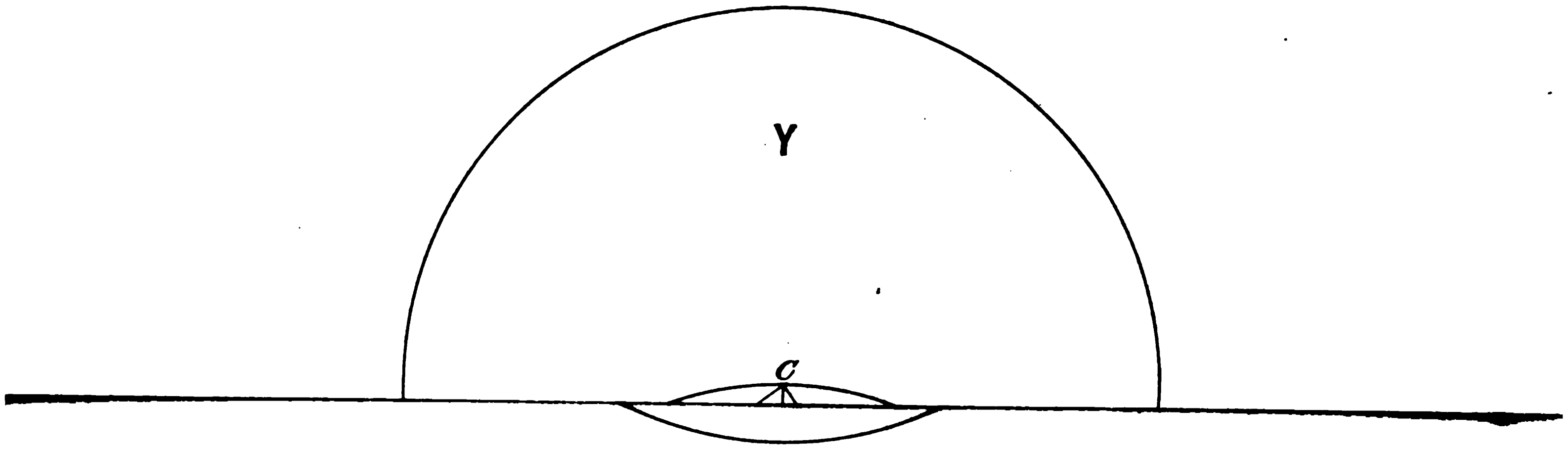
Hence:

A B : D B : : 8 : 1; that is, 77 : 9·625 : : 8 : 1.

And, $(AD^2 + DB^2) = (76·3960659^2 + 9·625^2) = (5836·3588499714281 + 92·640625) = 5928·9995099714281$, and is a very close approximation to $77^2 = 5929$.

Now, my dear Sir, I need not tell *you*, that the trigonometrical functions of angles are not lengths, but ratios of one length to another; but, I may tell *you*, that it is only when a right-angled triangle is *commensurable*, that the ratios of side to side can be given with arithmetical exactness. Well, then, the right-angled triangle A D B is *incommensurable*. But, the ratio of the side A B to the side D B is as 1 to ·125, or, as 8 to 1, and is arithmetically exact; but the triangle being incommensurable, we can only arrive at an approximation to the ratios between the sides A B and A D, and between the sides A D and B D; but, we can make this approximation as close as we please, by extending the number of decimals. With Logarithms to 7 places of decimals, the ratio of A B to A D is

DIAGRAM I.



as 1 to ·9921567, and the ratio of A D to B D is as ·9921567 to ·125; and these ratios are sufficiently accurate for all practical purposes.

Now, by Hutton's tables—and I take Hutton because he carries his logarithms to seven places of decimals—the natural sine of an angle of 7° 12′ is given as ·1253332, and the natural sine of an angle of 82° 48′ as ·9921147; the former *greater*, and the latter *less*, than their true value. I have been told by one of the first "*recognised Mathematicians*" of the day, that my *reasoning* on the ratio of diameter to circumference in a circle, will not "*stand the test of logic and common sense*:" but, I cannot help thinking that *you*, my dear Sir, will find it a very difficult matter to prove by "*the rules of logic and common sense*," either that the angle B A D, in the right-angled triangle A D B, is not an angle of 7° 12′, or, that ·125 is not the *true* value of the natural sine of this angle!!

You will now understand, that I maintain, not only that "*the received value of* π *is wrong*," but that the received "*trigonometrical expansion of tangent* ·x *is wrong*," and the received "*theory of series*" a fallacy.

In the enclosed Fig. (see Diagram I.), let A and B be two points dotted at random. Join A B. (Euclid: Post 1.) Produce A B to C, making A C equal to five times A B. (Euclid: Post 2.) With A as centre, and any interval greater than the half of A C, describe the circle X; and with C as centre, and the same interval, describe the circle Y. (Euclid: Post 3.) The circumferences of these circles intersect each other at the points *a* and *b*. Join these points. Then, the line *a b* bisects the line C A, at the point O. With O as centre, and O A or O C as interval, describe the circle Z. With C as centre, and C B as interval, describe the arc B D, and join C D and A D, producing the right-angled triangle C D A (Euclid: Prop. 31, Book 3). With A as centre, and A D as interval, describe the arc D E, and join E D, producing the triangle E A D, which *is* an *isosceles*, but not an *equilateral* triangle. From the angle A draw a straight line at right angles to C A, and therefore tangental to the circle Z, to meet a perpendicular let fall from the angle D in the triangle C D A, at the point G. Draw D F perpendicular to C A, and therefore parallel to G A, and join F G. From the point O, draw a straight line parallel to C D, to meet and terminate in the line A G at the point H, and join D H. The lines O H and

F D intersect each other at the point M. Join M A, producing the equilateral parallelogram M A H D. From the angle D, in the triangles C D A and C D F, draw a straight line through the point O, the centre of the circle Z, to meet and terminate in the circumference of the circle at the point K, and join C K and A K, producing the rectangular parallelogram C K A D; which is an inscribed rectangle to the circle Z. Produce C D to meet A G produced, at the point L. Produce *a b* to meet and terminate in the line C L, at the point P, and join P H and P A.

Now, my dear Sir, you will observe that I might have adopted a different method of construction. For instance: instead of saying with A as centre, and A D as interval, describe the arc D E: I might have said, with A as centre, and $\frac{3}{5}$ (A C) as interval, describe the arc E D; and you will see that the result would have been the same.

Now, C D = C B = $\frac{4}{5}$ (C A): and A D = A E = $\frac{3}{5}$ (C A), by construction. O H is parallel to C D, and therefore parallel to C L. K A is parallel to O H and C L. D G is parallel to C A, and D F is parallel to A L, and perpendicular to C A and D G. D A is parallel to C K, and at right angles to C D, and is therefore perpendicular to C L. C D A and C K A are similar and equal right-angled triangles, and C A, the diameter of the circle Z, is the hypothenuse of, and common to, the two triangles. All these facts arise out of the construction of this remarkable geometrical figure.

Let C A, the diameter of the circle Z, = 8.

Then . C D = K A = C B = $\frac{4}{5}$ (C A) = 6·4
D A = C K = A E = $\frac{3}{5}$ (C A) = 4·8
D L = $\frac{3}{4}$ (D A) = 3·6
C F = $\frac{4}{5}$ (C D) = 5·12
D F (G A) = $\frac{3}{5}$ (C D) = 3·84
F A = D G = (C A — C F) = $\frac{3}{5}$ (D A) = 2·88
G L = (A L — A G) = $\frac{3}{4}$ (D G) = 2·16
C L = $\frac{5}{4}$ (C A) = 10
O H = P A = P C = P L = $\frac{1}{2}$ (C L) = 5
O A = O C = $\frac{1}{2}$ (C A) = 4
A H = O P = $\frac{1}{2}$ (A L) = 3
H G = F M = (A G — A H) = ·84
F E = (A E — A F) = 1·92
C E = (C A — E A) = 3·2
P D = (C D — P A or P C) = 1·4

Then : by analogy or proportion :

C F : F D : : F D : F A ; that is, 5·12 : 3·84 : : 3·84 : 2.88.
and, C D : D A : : D A : D L ; that is, 6·4 : 4·8 : : 4·8 : 3·6.

Hence :

The sum of the squares of the four sides of the parallelogram F A G D = the sum of the squares of the diagonals F G and D A. The sum of the squares of the four sides of the parallelogram O A H P = the sum of the squares of the diagonals O H and P A. But, this equation = $3\frac{1}{8}(OA^2)$, that is to say, when the diameter of the circle Z = 8, then, $3\frac{1}{8}(OA^2) = 3\frac{1}{8}(4^2) = 3\cdot125 \times 16 = 50$. But, $3\frac{1}{8}(OA^2) = (OA^2 + AH^2 + OP^2 + PH^2)$, that is, $(3\cdot125 \times 16) = 16 + 9 + 9 + 16$, and this equation = 50 = area of the circle Z.

Now, O A H D is a quadrilateral, and the sum of the squares of the four sides = the sum of the squares of the four sides of the parallelogram O A H P ; that is, $(OA^2 + AH^2 + OD^2 + DH^2) = (OA^2 + AH^2 + OP^2 + PH^2)$, and this equation = area of the circle Z. But, "*in any quadrilateral the sum of the squares of the four sides is equal to the sum of the squares of the diagonals, together with four times the square of the line joining the middle points of the diagonals.*" Now, when the diameter of the circle Z = 8, then, $\frac{OA + AH}{10}$ $\frac{4 + 3}{10} = \frac{7}{10} = \cdot 7$, and this is the length of the line that would join the middle points of the diagonals in the quadrilateral O A H D. But, $4\left(\frac{OA + AH}{10}\right)^2 = 4(\cdot 7^2) = 4 \times \cdot 49 = 1\cdot 96$ DP^2 ; and it follows, that the sum of the squares of the diagonals in the quadrilateral O A H D + $DP^2 = 3\frac{1}{8}(OA^2)$, that is, $(OH^2 + DA^2 + DP^2) = 3\frac{1}{8}(OA^2)$; or, $(5^2 + 4\cdot 8^2 + 1\cdot 4^2) = 3\frac{1}{8}(4^2)$; or, $(25 + 23\cdot 04 + 1\cdot 96 = (3\cdot 125 \times 16)$, and this equation = 50 = area of the circle Z.

Again : M A H D is an equilateral parallelogram of which the sides = 3, and the diagonal D A = 4·8, when the diameter of the circle Z = 8 ; therefore, $\frac{DA}{2} = 2\cdot 4 = Dx$, and DxM is a right-angled triangle ; therefore, $DM^2 - Dx^2 = 3^2 - 2\cdot 4^2 = 9 - 5\cdot 76 = 3\cdot 24 = Mx^2$; therefore, $\sqrt{3\cdot 24} = 1\cdot 8 = Mx$; therefore, $2(Mx) = 3\cdot 6$ = the diagonal M H. Hence : the sum of the squares of the diagonals = the sum of the squares of the four sides of the parallelogram M A H D, that is, $(4.8^2 + 3\cdot 6^2) = 4(3^2)$, or, $(23\cdot 04 + 12\cdot 96) =$

(4 × 9) and this equation = 36, and is equal to a square on A L the base of the right-angled triangle C A L.

Now, D F O is a right-angled triangle, and D O is a radius of the circle Z = 4, and D F = 3·84; therefore, $(DO^2 - DF^2) = (4^2 - 3{\cdot}84^2) = (16 - 14{\cdot}7456) = 1{\cdot}2544 = FO^2$; therefore, $\sqrt{1{\cdot}2544} = 1{\cdot}12 = FO$. But, $\frac{FO}{DO} = \frac{1{\cdot}12}{4} = {\cdot}28$, and this is the natural sine of the angle O D F: and, $\frac{DF}{DO} = \frac{3{\cdot}84}{4} = {\cdot}96$, and this is the natural sine of the angle D O F. Again: A D P is a right-angled triangle, and the side A P = 5, the side D P = 1·4, and the side A D = 4·8. Hence: $\frac{DP}{AP} = \frac{1{\cdot}4}{5} = {\cdot}28$: and, $\frac{AD}{AP} = \frac{4{\cdot}8}{5} = {\cdot}96$. Again: A F M and D G H are similar and equal right-angled triangles. Take the triangle D G H. The side D G = 2·88; the side G H = ·84; and the side D H = 3. Hence: $\frac{GH}{DH} = \frac{{\cdot}84}{3} = {\cdot}28$, and $\frac{DG}{DH} = \frac{2{\cdot}88}{3} = {\cdot}96$. Hence: the triangles D F O, A D P, A F M, and D G H, are similar right-angled triangles, and have the sides that contain the right angle in the ratio of 7 to 24; the hypothenuse to perpendicular in the ratio of 25 to 7; and the hypothenuse to base in the ratio of 25 to 24. In all these triangles the acute angle is an angle of 16° 16′, and the natural sine = ·28, and the obtuse angle is an angle of 73° 44′, and the natural sine = ·96. You may readily convince yourself that these are the true arithmetical values of the natural sines of the angles, and not ·2801083 and ·9599684, as given in Hutton's Tables.*

Again: The triangles A O P and A D L are similar right-angled triangles, and when the diameter of the circle Z = 8, A O = 4: O P = 3: A P = 5: A D = 4·8: D L = 3·6: and A L = 6. Hence: $\frac{OP}{AP} = \frac{DL}{AL}$, that is, $\frac{3}{5} = \frac{3{\cdot}6}{6} = {\cdot}6$, and this is the natural sine of the angles O A P and D A L: and, $\frac{OA}{AP} = \frac{AD}{AL}$, that is, $\frac{4}{5} = \frac{4{\cdot}8}{6} = {\cdot}8$, and this is the natural sine of the angles A P O and

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* It is self-evident that—geometrically—the natural sines of the angles in the triangles D F O, A D P, and A F M are all different. Hence, ·28 and ·96 are not the arithmetical values of the *natural sines*, but of the *trigonometrical sines* of the acute and obtuse angles.

A L D. The acute angle in these triangles is an angle of 36° 52′, and the obtuse angle an angle of 53° 8′—and again I say, you may readily convince yourself that ·6 and ·8 are the true arithmetical values of the natural sines of these angles, and not ·5999549 and ·8000338 as given in Hutton's Tables. I need not point out the numerous right-angled triangles in this remarkable geometrical figure, which have the sides that contain the right angle in the ratio of 3 to 4.

Now, it is self-evident, that the angles O A P, P A D, and D A L, are together equal to the right angle O A L. If it be said that this is not self-evident, the proof is very simple. We have only to describe an arc from a point in C A to a point in A L, with A as centre and A P as interval, thus describing a quadrant of a circle, of which the angle A would be an angle at the centre = 90° Hence: the angles O A P, P A D, and D A L = (36° 52′ + 16° 16′ + 36° 52′), are together equal to a right angle = 90°.

I must now direct your *especial* attention to a right-angled triangle in this remarkable figure, which has no companion; that is to say, there is no similar triangle in the figure. This is the triangle D F E, of which the sides that contain the right angle are in the ratio of 2 to 1. Now, when the diameter of the circle Z = 8, the side D F = 3·84, and the side F E = 1·92; therefore, $DF^2 + FE^2 = (3{\cdot}84^2 + 1{\cdot}92^2) = (14{\cdot}7456 + 3{\cdot}6864) = 18{\cdot}432 = DE^2$; therefore, $\sqrt{18{\cdot}432} = 4{\cdot}29325 = DE$, approximately; and the triangle is an incommensurable right-angled triangle. But, $\frac{FE}{DE} = \frac{1{\cdot}92}{4{\cdot}29325} = {\cdot}4472136$, and this is the natural sine of the angle F D E: and, $\frac{FD}{DE} = \frac{3{\cdot}84}{4{\cdot}29325} = {\cdot}8944272$, and this is the natural sine of the angle D E F. The angle F D E is equal to half the angle $FDC = \frac{53^\circ\ 8'}{2} = 26^\circ\ 34'$, and makes the angle D E F, and angle of 63° 26′.

Well, then, it is perfectly obvious, that in a right-angled triangle of which the sides that contain the right angle are in the ratio of 2 to 1; the arithmetical value of the natural sine of the obtuse angle must be the double of the arithmetical value of the natural sine of the acute angle. No angle answering to this is to be found in tables of natural sines, and it follows, by "*the rules of logic and common sense*," that these tables are *fallacious*. You will find that an angle of 26° 34ʹ, approximates very nearly to this requirement, the natural sine and co-sine being given as ·4472388 and ·8944146.

Now, by hypothesis, let CA the diameter of the circle $Z = 1$. Then: $\frac{C\,A}{2} = \frac{1}{2} = {\cdot}5 = O\,A =$ radius. Hence: $\frac{25}{24}(O\,A) = \frac{25 \times {\cdot}5}{24} = \frac{12{\cdot}5}{24} =$ ·5208333333 with 3 to infinity. But, 6 times ·5208333333 = 3.124999998, and "*represents to us the determinate arithmetical value*" 3·125, as certainly as the never ending series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} +$, &c., "*represents to us the determinate arithmetical value* 2." Well, then, this makes 3·125 the true arithmetical value of π. Hence: $\frac{24}{25}(3{\cdot}125) = \frac{24 \times 3{\cdot}125}{25} = 3$, and it follows of necessity, that $\frac{24}{25}$ and $\frac{3}{3{\cdot}125}$ are equivalent ratios, and both express the ratio between the perimeter of every regular hexagon and the circumference of its circumscribing circle.

Now, D F C is a right-angled triangle, and D E C a part of it, is an oblique-angled triangle. Hence: $\{D\,E^2 + E\,C^2 + 2\,(E\,C \times E\,F)\} = C\,D^2$; that is, $\{18{\cdot}432 + 10{\cdot}24 + 2(3{\cdot}2 \times 1{\cdot}92)\} = 6{\cdot}4^2$; or, $(18{\cdot}432 + 10{\cdot}24 + 12{\cdot}288) = 40{\cdot}96$. Again: D F C is a right-angled triangle, and D F O a part of it, is an oblique-angled triangle. Hence: $\{D\,O^2 + O\,C^2 + 2\,(O\,C \times O\,F)\} = D\,C^2$, that is, $\{4^2 + 4^2 + 2\,(4 \times 1{\cdot}12)\} = 6{\cdot}4^2$: or, $(16 + 16 + 8{\cdot}96) = 40{\cdot}96$. Again: A D C is a right-angled triangle, and A P C a part of it, is an oblique-angled triangle. Hence: $\{A\,P^2 + P\,C^2 + 2\,(P\,C \times P\,D)\} = C\,A^2$; that is, $\{5^2 + 5^2 + 2\,(5 \times 1{\cdot}4)\} = 8^2$; or, $(25 + 25 + 14) = 64$. Again: D G A is a right-angled triangle, and D H A a part of it, is an oblique-angled triangle. Hence: $\{D\,H^2 + H\,A^2 + 2\,(H\,A \times H\,G)\} = D\,A^2$: that is, $\{3^2 + 3^2 + 2\,(3 \times {\cdot}84)\} = 4{\cdot}8^2$: or, $(9 + 9 + 5{\cdot}04) = 23{\cdot}04$. Again: the triangle D F C and D F A on each side of D F, are similar to the whole triangle C D A, and to each other. Again: the triangle A D C and A D L on each side of A D, are similar to the whole triangle C A L, and to each other. Again: the triangles D G A and D G L on each side of D G, are similar to the whole triangle A D L, and to each other.

Now, my dear Sir, you cannot fail to perceive, that from the properties of this remarkable geometrical figure, we can reason—"*soundly and logically*"—that Euclid is *not* at fault in the 12th proposition of the second book, and 8th proposition of the sixth book: and yet, it may be readily demonstrated, that *Euclid is at fault*, in attempting to make these propositions of general and universal application, and therefore *true* under all circumstances.

In conclusion, I may observe:—Without putting you to the inconvenience of purchasing any of my publications, I cannot help thinking that in this communication I have given sufficient information to convince you, that neither by the *series* $\frac{16}{\sqrt{3}}\left\{\frac{1}{1 \cdot 3} + \frac{2}{5 \cdot 7} \cdot \frac{1}{9} + \frac{3}{9 \cdot 11}\left(\frac{1}{9}\right)^2 + \&c. \right\}$, nor any other series, can we arrive at either the *true* arithmetical value of π, or the *true* arithmetical value of the natural sine of an angle.

Believe me,
My dear Sir,
Very truly yours,
JAMES SMITH.

PROFESSOR W. ALLEN WHITWORTH,
Queen's College.

THE REV. PROFESSOR WHITWORTH *to* JAMES SMITH.

LIVERPOOL, *November* 28*th*, 1868.

MY DEAR SIR,

I am surprised that you should write on page 4, of your Letter, $\frac{1}{7} = \cdot 142857$, $\frac{2}{7} = \cdot 285714$, &c., and then add up these results, as if the decimals stopped after six places. Of course we can get no true results, unless we recognise the fact that the periods of decimals continue *ad infinitum.* The same remark applies to reasoning on p. 3, about the fractions $\frac{1}{17}$, $\frac{2}{17}$, $\frac{3}{17}$ ......

I perceive by your observation, on page 5, that you do not realize what we mean by a series tending to a limit. Neither of the series which you name, has the property which you attribute to both, viz.: that by an extension of the number of terms they can be made to tend towards an integral finite quantity, and differ from that finite quantity by less than any assignable quantity. You seem to speak indifferently of the series (α) $\frac{1}{7} + \frac{2}{7} + \frac{3}{7} + \ldots$ and of the smaller series (β) ·142857 + ·285714 + ·428571 + ... You would be quite right in asserting that 7 terms of (α) come to 4, that 14 terms come to 15, that 21 terms come to 33, that 28 terms come to 58, and so on. But you must use the word *tend* in a different sense from that in which I use it, if you say that the sum of this series (α) tends to anything but ∞.

But it is probably to the series (β) that you intend the assertion on page 5 to apply. But then this series does not tend to *a* finite quantity. All you can mean is, that by taking a certain fixed number of terms, you can make the sum come *very nearly* to certain different integral sums. But I simply do not understand the assertion, that the sum can be made to differ from such a finite quantity, by less than any assignable quantity. For instance, if I assign the quantity $\frac{1}{10^{20}}$, can you by extending the number of terms of the series (β) make the sum differ from 4, by less than the assigned quantity $\frac{1}{10^{20}}$?

The case is quite different with such a series as I instanced. I say that the sum of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ + &c., tends to the limit 2, and can be made to differ from 2 by less than *any assignable quantity*, by sufficiently increasing the number of terms. For instance, if you assign the quantity $\frac{1}{10^{20}}$ you have only to take 61 terms of the series, and the sum of those 61 terms will differ from 2 by less than $\frac{1}{10^{20}}$, and similarly for any other assignable difference.

Of course you see that as $\frac{1}{7}$ is not equal to ·142857, the two series (α) and (β) are totally distinct series; and the series (γ) ·142857142857 + ·285714285714 +, &c., would again be an entirely different series from either of the others.

I again repeat, that if the value of π, and the theory of series be upset, the calculations of eclipses are upset also. It is true that the rough periods of 18 years 11 days, and the like, have been gathered from observation; but the accurate calculations of the *exact second* at which an eclipse occurs depend upon mathematical reasoning, which must be abandoned if you upset π and the theories of series.

With respect to your reasoning on page 7, I would point out that you cannot infer, that, because two numbers have the same Logarithm to 20, or to 100 decimal places, therefore they are equal. For the two numbers may differ by a quantity so small as only to affect the 101st decimal place in the Logarithm, and yet sufficient to destroy equality between the numbers.

I am much obliged to you for giving me, on pages 8 and 9 of your Letter, what you consider a proof that $\pi = \frac{25}{8}$. You, however, unfortunately assume your result, and merely shew that it leads to a true conclusion: and YOU SAY, that we cannot get these equations by any other value of π. If you can *prove* this last assertion, it may complete your demonstration, but, of course, if any other assumed value of π will give the same identities at last, the claim of $3\frac{1}{8}$ falls to the ground.

Now, it appears to me, that if you will assume $\pi =$ any fraction whose numerator and denominator are powers of 5 and 2, the same identities will be obtained. For instance, we may write $\frac{16}{5}$ instead of $\frac{25}{8}$ all through your work, and the result holds good. Or the same "*proof*" will equally apply to $\pi = \frac{512}{125}$. May I ask whether you tried such numbers as these before you asserted that $\pi = 3{\cdot}125$ was the only hypothesis that would lead to the true result you gained?

Indeed, ANY value assumed for π would satisfy your argument on pages 8 and 9, if we were to work algebraically. The only reason why $\frac{22}{7}$, or $\frac{355}{113}$, or $3{\cdot}14159$ SEEM to fail is, that these numbers themselves, or else their reciprocals when expressed *arithmetically in decimal notation*, have an infinite series of figures in their expansion, and you only take a few of them.

I give you the credit of desiring only to arrive at the truth in this matter; you will therefore not think me ill-natured in pointing out the fallacy of your reasoning—I feel sure that, on the contrary, you will thank me. I, for my part,—if it be true that $\pi = 3{\cdot}125$—should be glad to arrive at that truth. But, as yet, I have found no flaw in the argument by which it is shewn that π has the value defined in my last Letter, nor have I met with an argument in favour of any other.

Looking at pages 14 and 15, I find that you prove, at considerable length, that (1) when $\pi = \frac{25}{8}$, and (2) when $\pi = 3{\cdot}1416$, we have the following equation :—

$$\frac{\text{Circular measure of } 90^\circ}{\text{Circular measure of } \frac{1}{25}\text{th of } 360^\circ} = 6{\cdot}25.$$

No one would have doubted this, since the numerator and denominator of the first fraction, are evidently in the ratio 25 : 4 ; and therefore, (whatever assumption be made about π,) the fraction $= \frac{25}{4} = 6{\cdot}25$. But you observe, and truly, that, on your hypothesis, this number 6·25 expresses the circumference of a circle whose radius is unity ; and, on the orthodox theory, it does not. But, you have not shewn value. When you have tried $\pi = \frac{16}{5}$, and found it to answer all the purposes to which you apply $\pi = \frac{25}{8}$, in your Letter ; you will perceive that your argument, ingenious as it is, really proves nothing.

I now pass on to the next argument you put forth. You refer me to a pamphlet, of which you kindly forward a copy ; and you quote a Letter from a "recognised Mathematician" (recognised by whom ? ? ?) who says the demonstration on pp. 14 and 15 ought to have sufficed.

Why should it be expected that $\dfrac{\text{Circular measure of } 90^\circ}{\text{Circular measure of } \frac{4}{25} \text{ of } 90^\circ}$ should represent the circumference of a circle of radius unity ? We cannot draw an inference in favour of either theory, and all that is proved is this : *that the two theories do not agree*, which I think we knew before.

Perhaps it will suffice to stop here for the present, and leave the discussion of your other arguments till you have considered the objections to these.

Of course a plurality of proofs is needless. If you can give *one* proof that $\pi = \frac{25}{8}$, we shall be satisfied.

I would point out, in conclusion, that as our question is not about the *approximate* value of π, but its *absolute* value ; no satisfactory arguments can be drawn from elaborate calculations which assume the APPROXIMATE values of logarithms, and of trigonometrical ratios. At least, you must not use these approximate values without calculating *to what degree* the error of the approximation will affect the accuracy of the result.

As I find that, in the latter part of your Letter, you quote Euclid, may I enquire, before I study that part of your argument, whether you have given up the idea that "Euclid is at Fault;" or, if not, where his fault begins? You assume Prop. 31, Book 3. Shall you object to my assuming, in reply, any proposition previous to this one?

I hope you are not going to drag my name into a pamphlet, as I don't like notoriety. But if you do publish any of the Letters which you have addressed to me, giving my name, I should wish this Letter, in its integrity, to accompany them. I should, however, much prefer to escape the publicity which I see you have given to previous Correspondents, as my only wish is to convince—or to be convinced—(as the case may be) of the TRUTH, and not to establish a tournament for the entertainment of others.

I am, Sir, faithfully yours,
W. ALLEN WHITWORTH.

P.S.—I thank you for your kind invitation, but I at present arrange not to go out to Seaforth. W. A. WH.

JAMES SMITH *to* THE REV. PROFESSOR WHITWORTH.

BARKELEY HOUSE, SEAFORTH,
Saturday Evening,
28*th November*, 1868.

MY DEAR SIR,

I beg to thank you for the promptitude with which you have replied to my Letter dated 23rd inst.,—posted yesterday—which could only have come into your hands this morning.

I *assume* that your knowledge of my pamphlet, "Euclid at Fault," is limited to the quotation taken from the *Liverpool Leader*, of 19th September, 1868, and given in your original communication, from which you inferred that I am a *maligner* and *libeller* of "*recognised Mathematicians*."

I send you, herewith, a copy of "Euclid at Fault," and if you compare the diagram in *it*, with that enclosed in my last communication, it will be self-evident to you, that both contain a rectangular parallelogram within a circle, having all the angles touching the circumference: and I put the following questions to you:—

Can dis-similar rectangular parallelograms be inscribed in circles of the same diameter, and have all the angles touching the circumferences?

When the diagonals of the rectangular parallelograms, in the two diagrams, are represented by the arithmetical symbol 8, what are the arithmetical values of the sides of the parallelograms?

Waiting the favour of your answers to these two questions,

Believe me, my dear Sir,

Faithfully yours.

JAMES SMITH.

THE REV. PROFESSOR WHITWORTH.

THE REV. PROFESSOR WHITWORTH *to* JAMES SMITH.

LIVERPOOL, *November 28th*, 1868.

MY DEAR SIR,

Continuing my examination of your Letter received this morning, I now come to your geometrical construction in pages 15, et seqq.

When I come to page 19, I find that you *assume* $\pi = 3\frac{1}{8}$. Consequently the deductions from your argument can not be taken as *proving* $\pi = 3\frac{1}{8}$—you argue in a circle.

However, none of your results are very startling, till we get to page 21, where you assume, without proof (and I should say you wrongly assume,) that certain acute angles in your figure are 16° 16′ exactly. Pray how do you arrive at this conclusion? Is not the angle a very small fraction less than this? I will grant that the sines of your angles are ·28 and ·96, but I deny that the angles are exactly 16° 16′ and 73° 44′.

But the most curious instance of false reasoning, which I have met with, is on page 23. You shew that two angles must have their sines in the ratio of 2 to 1 : and then you observe that no angles, satisfying the required conditions are found in the published tables : whence you reason that the tables must be wrong, forgetting that the tables only register the sines and co-sines of angles which contain ***an integral number*** of seconds (or it may be, minutes). The only conclusion therefore which "*the rules of logic and common sense*" can lead you to is this, that the angles in question do not contain an integral number of seconds (or minutes). If I had any tables here, I could readily calculate to the decimal of a second what the angle must be. You will probably find it VERY LITTLE less than 26° 34′. (Its circular measure = $\frac{1}{2} - \frac{1}{24} + \frac{1}{160} - \frac{1}{896} + \frac{1}{4608} - \frac{1}{22528} + \ldots\ldots$, *ad infinitum.*)

I am sorry to have to object to one of your statements on page 24, when you say that "3·124999998 represents to us the determinate arithmetical value 3·125," *which is as much as to say that* ·000000002 = 0. ! ! ! But of course I should be ready to admit that 3·12499999...... (where the dots indicate that the 99... are to be continued for ever) ; or, $3{\cdot}124\dot{9}$ (where the mark over the 9 indicates that the 9 is to be repeated for ever) represents to us the determinate arithmetical value 3·125, as certainly as the never ending series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$ &c., represents to us the determinate arithmetical value 2, (where the "&c." indicates that the series is to be continued, *ad infinitum.*)

Finally, with reference to the paragraph at the foot of page 25, I must observe that, as Euclid's reasoning is perfectly general, his results in the propositions you name are true universally, unless there be a flaw in his reasoning. If there is a flaw you can doubtless point it out. If the propositions are not true universally, they are not true as Euclid states them, and therefore Euclid is at fault. But I am familiar with every step in Euclid's proofs, and have satisfied myself of the soundness of his arguments. I cannot see that when we eliminate the errors you have fallen into, which I have pointed out, there will remain anything in your Letter to justify any hesitation in accepting the orthodox value of π, which is both finite and definite ; or to lead us to doubt the truth of the propositions to which you object, in the Second and Sixth books of Euclid's Elements.

However, as you think that Mathematicians have treated you with impatience and discourtesy, I shall be ready to read, with the utmost patience and the most courteous consideration, anything further that you may have to allege in favour of your views; and shall also be happy to explain myself more fully, if you are unable to recognise the fallacies which I have attempted to point out.

Will you please to consider this as a continuation of my Letter of this morning.

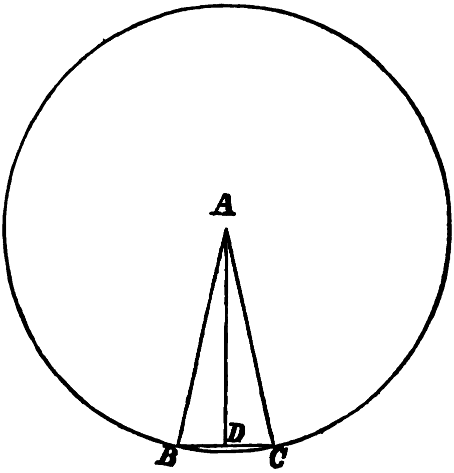
And believe me, my dear Sir,

Very truly yours,

JAMES SMITH, ESQ. W. ALLEN WHITWORTH.

P.S.—I omitted to observe, with respect to your argument on page 14, that your own result is sufficient to prove that π *is not* $\frac{25}{8}$.

I adopt your construction of page 10, so that A is the centre of a circle in which is described a regular xxv.-gon, of which one side B C is bisected in D. You say, page 13, A B : B D = 1 : ·125 = 8 : 1; or, B D = $\frac{1}{8}$th of radius A B; therefore, B C, the double of B D, must be $\frac{1}{4}$th of the radius, or $\frac{1}{8}$th of the diameter, *i.e.*, chord B C = $\frac{1}{8}$th of diameter.



But, arc B C = $\frac{1}{25}$th of circumference.

= $\frac{1}{25}$th of $\frac{25}{8}$th of diameter—
on Mr. Smith's hypothesis

= $\frac{1}{8}$th of diameter.

Therefore, Chord B C = Arc B C.

Or the straight line is not the shortest distance between the two points B and C!!!

I should think that this must satisfy you that you have been misled. W. A. WH.

THE REV. PROFESSOR WHITWORTH *to* JAMES SMITH.

LIVERPOOL, *November 30th*, 1868.

MY DEAR SIR,

I am very much obliged to you for your copy of "Euclid at Fault;" for, though I read the Pamphlet some months ago, it may be convenient to possess a copy, to refer to in the present correspondence.

I do not, however, quite understand whether you still hold the views expressed in the Pamphlet, or whether I am to infer, from your quotation of Euclid iii. 31, that your confidence in Euclid's soundness is restored. Of course, if he is wrong in the Second Book, or in the Sixth, there must be a flaw in his reasoning somewhere, and his objectors ought to point it out. If he is wrong in Book ii., of course we cannot accept the results which he deduces in Book iii.

You ask two simple questions in your Letter received this day. Whether you ask them in order to examine me, or for any other purpose, I have no objection to give you the answers which any Mathematician must give.

To the first question, we say that any number of dis-similar rectangles (*i.e.*, "rectangular parallelograms") can be inscribed in circles of the same diameter, having all the angles touching the circumference.

The second question falls to the ground, when the first is answered in the affirmative. For, as you can describe a thousand such rectangles, there is no limit to the number of different values which the sides of the parallelograms may have. I will give you as many as you please; and, indeed, the *Leader* has already answered an equivalent question, in the Article you refer to.

Taking, as you suggest, a circle whose diameter is 8 units of length, the following will express the lengths of the sides of various dis-similar rectangular parallelograms, which may be inscribed in it.

	Length.	Breadth.	Area.
1st Rectangle	6·4	4·8	30·72
2nd	7·68	2·24	17·2032
3rd	$7\frac{5}{13}$	$3\frac{1}{13}$	
4th	$7\frac{1}{17}$	$3\frac{13}{17}$	
5th	$7\frac{21}{37}$	$3\frac{15}{37}$	
6th	$\frac{320}{41}$	$\frac{72}{41}$	
7th	$\frac{480}{61}$	$\frac{88}{61}$	

And so on, as many as you please. Or we may get the following, of which the ratio of the sides is incommensurable ; but which can be constructed geometrically with *perfect* accuracy.

	Length.	Breadth.	Area.
8th Rectangle	$3\sqrt{7}$	1	$3\sqrt{7}$
9th	$2\sqrt{15}$	2	$4\sqrt{15}$
10th	$\sqrt{55}$	3	$3\sqrt{55}$
11th	$4\sqrt{3}$	4	$16\sqrt{3}$
12th	$\sqrt{39}$	5	$5\sqrt{39}$
13th	6	$2\sqrt{7}$	$12\sqrt{7}$
14th	7	$\sqrt{15}$	$7\sqrt{15}$

Or you may take the following, in which the sides are commensurable with one another, but incommensurable with the unit of length ; and the area is commensurable with unity.

	Length.	Breadth.	Area.
15th Rectangle	$\frac{16}{5}\sqrt{5}$	$\frac{8}{5}\sqrt{5}$	$\frac{128}{5}$
16th	$\frac{12}{5}\sqrt{10}$	$\frac{4}{5}\sqrt{10}$	$\frac{96}{5}$
17th, the Square	$4\sqrt{2}$	$4\sqrt{2}$	32

Any number of these may be determined at once. But probably I have supplied you, in these seventeen, with sufficient instances of rectangles dis-similar to one another, but all inscribable in a circle of radius 4.

And now that I have answered, in all fulness, the questions which I had the honour to refer to you, may I recall your attention to the subject of my previous Letters, and ask you to mention if there is anything illogical or unconclusive in my review (in my Letters of Saturday last) of your arguments on the subject of the value of π.

And if you recognise and appreciate the correctness of my strictures, on the arguments you have adduced, will you point me to any other arguments which you can bring forth to prove $\pi = \frac{25}{8}$. If you can give me one single proof (in which I can detect no flaw) that your value of π is correct, I will then scrutinise once more the proofs by which the orthodox value of π is established. Of course, both results cannot be right, and therefore either your argument or the argument of recognised Mathematicians must be unsound. I am quite ready to go on and find where this unsoundness lies. If you can give me a proof in which I see nothing illogical or unsound, I will immediately publish it in the *Mathematical Journal* * which I edit. If, on the contrary, I am able to point out something defective in any demonstration you may present, I think it will then, at least, be your part either to shew where is the fault in the orthodox proof, or else to acknowledge yourself in the wrong.

Believe me, dear Sir,
Faithfully yours,
W. ALLEN WHITWORTH.

JAMES SMITH, ESQ.,
Seaforth.

JAMES SMITH *to* THE REV. PROFESSOR WHITWORTH.

BARKELEY HOUSE, SEAFORTH,
30th November, 1868.

MY DEAR SIR,

You will see the importance of the questions I put in my short Letter of the 28th instant; for, since $HO^2 + OK^2 + 2(OK \times OF) = HK^2$, in the diagram in "*Euclid at Fault;*" that is to say, when the radius of the circle = 4, then, $HO^2 + OK^2 + 2(OK \times OF) = 40$, it follows, that if the lines C D and K A, sides of the rectangular parallelogram in the diagram enclosed in my Letter of the 23rd instant, can be proved to be $\sqrt{40}$, when C A the diameter of the circle Z = 8, that Euclid is *not* at fault, and that I am altogether wrong.

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* "The Oxford, Cambridge, and Dublin Messenger of Mathematics; a Journal supported by Junior Mathematical Students of the Three Universities."

I cannot pretend to answer your Letter paragraph by paragraph, but in one part of it you say :—" *Now, it appears to me, that if you will assume* π = *any fraction, whose numerator and denominator are powers of* 5 *and* 2, *the same identities will be obtained. For instance, we may write* $\frac{16}{5}$, *instead of* $\frac{25}{8}$, *all through your work, and the result holds good; or the same will equally apply to* $\pi = \frac{512}{125}$." This I must respectfully deny; these are mere assertions, without a shadow of proof. You then put the question :—*May I ask whether you tried such numbers as these, before you asserted that* π = 3·125 *was the only hypothesis that would lead to the true result you gained?*" I certainly never attempted anything so absurd as to assume such a value of π as $\frac{512}{125}$, which would make the value of π greater than the perimeter of a circumscribing square to a circle of diameter unity. But, I can show you some important consequences resulting from assuming $\frac{16}{5} = 3\cdot2$ as the value of π. In the analogy or proportion, A : B : : B : C, when A denotes $\frac{3\cdot2}{4}$ and B denotes 1 ; then, ·8 : 1 : : 1 : ·125, and the product of the means is equal to the product of the extremes. Now, if the radius of a circle = ·125, then, (6 × ·125) = ·75 = the perimeter of a regular inscribed hexagon ; and 3 : 3·125 : : ·75 : ·78125. Hence: $\frac{3}{3\cdot125}$ and $\frac{\cdot75}{\cdot78125}$ are equivalent ratios, and both express the ratio between the perimeter of every regular hexagon and the circumference of its circumscribing circle. Again: 3·2 (10 × ·125) = 3·2 × 1·25 = 3·90625. Hence : 3·90625 : 1·25 : : 3·125 : 1, and it follows, that $\frac{3\cdot90625}{1\cdot25}$ and $\frac{3\cdot125}{1}$ are equivalent ratios, and both express the ratio between the circumference and diameter in every circle.

Take the following example of continued proportion :—

A : B : : B : C : : C : D : : D : E.
Let A = 4 and B = 5.

Then

C = 6·25 : D = 7·8125 : and E = 9·765625 = $3\cdot125^2 = \pi^2$: and I must leave you to follow out the consequences.

I am afraid you have read my Letter of the 23rd inst. too hastily, for there are several points in your reply in which you "*misinterpret*" me.

A short time ago I received a Letter, of which the following is a copy. The writer is a well known author on Scientific subjects :—

LONDON, 17*th October*, 1868.

MY DEAR SIR,

I am to give a Lecture shortly, at the Crystal Palace, where they have Free Lectures once or twice a month during the season now commencing. As my subject takes in Astrology, Alchemy, Squaring the circle, &c.—I should like to shew the latest offered solution of the latter problem. Now, if it would be agreeable to you, I would put up, among my illustrations, an enlarged diagram of your method—and read a short—very short notice of it—for I should necessarily be obliged to be brief.

I simply place these matters before my audience in an historical point of view, and shew that, to the present time, they are none of them wholly extinct.

I saw very little of you at Norwich, but the fact is you had such a *learned* and lively coterie at the Royal Hotel, that we of the Norfolk were quite thrown into the shade!

Hoping this will find you in good health,

I am, Yours very truly,

D.

The following is a copy of my reply, which is plain and simple enough, and is based on facts admitted by Mathematicians, and ought to be convincing that the problem of squaring the circle is "*un fait accompli.*"

BARKELEY HOUSE, SEAFORTH,
19*th October*, 1868.

MY DEAR SIR,

I am in receipt of your much esteemed favour of the 17th inst.

In that communication you inform me, that you are to give a Lecture at the Crystal Palace shortly, and observe :—"*As my* *subject takes in Astrology, Alchemy, Squaring the Circle, &c., I should like to shew the latest offered solution of the latter problem.*" You also say: "*Now, if it would be agreeable to you, I would put up, among my illustrations, an enlarged diagram of your method.*

Now, my dear Sir, my methods are manifold, but I can give you, "*the latest offered solution*" of the problem of "*Squaring the Circle,*" and am vain enough to think that I can make it a labour of love to you, to refer to it in your Lecture.

The following geometrical theorem was long ago discovered, and known to Geometers and Mathematicians.

"*In any quadrilateral the sum of the squares of the four sides, is equal to the sum of the squares of the diagonals, together with four times the square of the lines joining the middle points of the diagonals.*"

Now, if I were to say that this theorem is not true, Professor de Morgan might very properly say of me:—*James Smith, Esq., of Liverpool, is nailed by himself to the barn-door, as the delegate of miscalculated and disorganised failure.*" * But, had that learned gentleman been "*a reasoning geometrical investigator*"—which he says I am not—he would long ago have made this theorem the means of "*Squaring the Circle*:" or, in other words, the means of discovering the true ratio of diameter to circumference in a circle.

I construct the enclosed figure *(See Diagram II.)* in the following way. I draw two straight lines at right angles, making O the right angle. From the point O in the direction O A, I mark off four equal parts together equal to O A; and from O in the direction O B, I mark off three of such equal parts, together equal to O B, and join A B. It is obvious or self-evident, that A O B is a right-angled triangle, of which the sides that contain the right angle are in the ratio of 4 to 3: by construction. With A as centre and A B as interval, I describe the circle X, produce A O and B O to meet and terminate in the circumference of the circle at the points G and C, and join A C, C G, and B G, producing the quadrilateral A C G B. I bisect A G at F, and with O as centre, and O F as interval, describe the circle Z. The line O F is the line that joins the middle points of the diagonals in the quadrilateral A C G B; and it follows, that, $\{A G^2 + C B^2 +$

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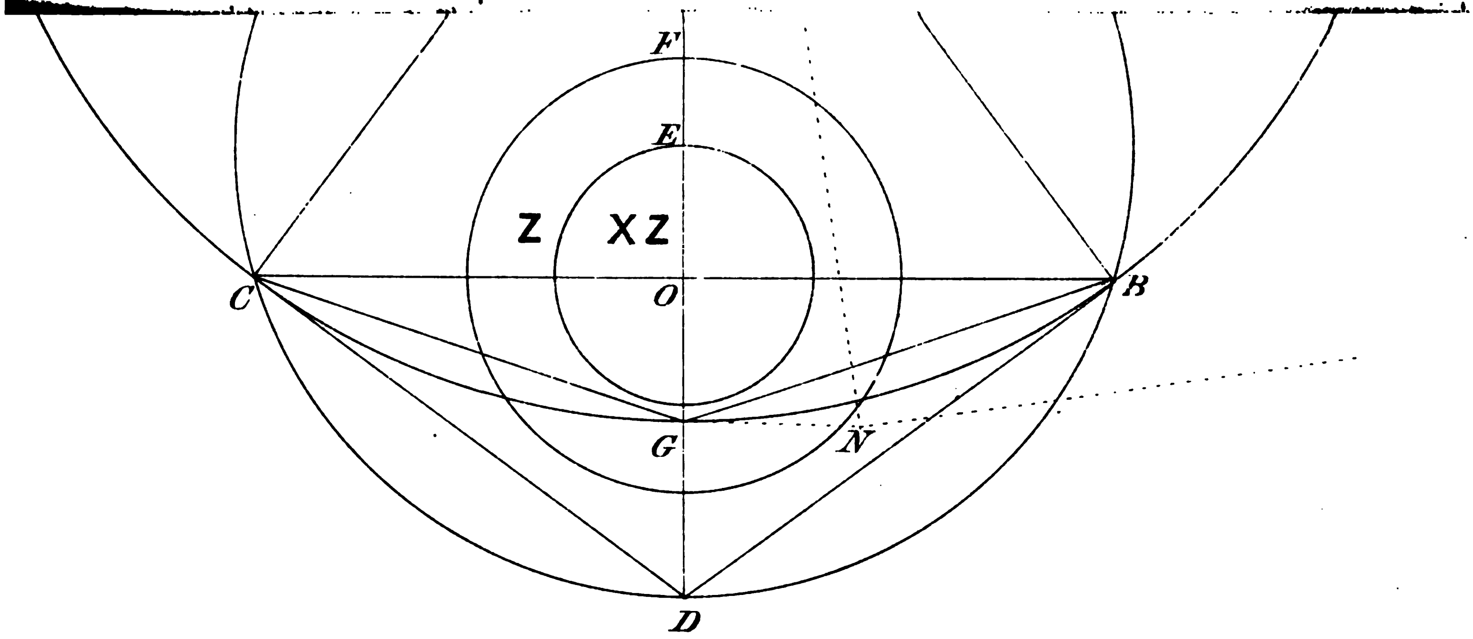
* *See Athenæum*: 25th July, 1868. Article; Our Library Table.

DIAGRAM II.



$4 (OF^2)\} = \{AC^2 + CG^2 + BG^2 + AB^2\}$. When $AO = 4$, we get the following equation $\{5^2 + 6^2 + 4 (1{\cdot}5^2)\} = \{5^2 + \sqrt{10}^2 + \sqrt{10}^2 + 5^2\}$, or, $(25 + 36 + 9) = (25 + 10 + 10 + 25) = 70$. From the points B and C I draw straight lines at right angles to A B and A C, and therefore tangental to the circle X, to meet A G produced at D, and join B D and C D, producing the quadrilateral A C D B. I bisect A D at E, and with O as centre and O E as interval, describe the circle X Y, and with E as centre and E A or E D as interval, describe the circle Y.

By Euclid: Prop. 31: Book 3: the triangles A B D and A C D are right-angled triangles, and it is self-evident that A D is the hypothenuse of, and common to, the two triangles. It is also self-evident, that the triangles on each side of A O are similar and equal triangles; and the triangles on each side of O D are also similar and equal triangles. It is also self-evident that the triangles on each side of B O and C O are right-angled triangles, and the sides B O and C O perpendicular to A D, the diameter of the circle Y. Hence: By Euclid: Prop. 8: Book 6: the triangles on each side of B O are similar to the whole triangle A B D, and to each other: and the triangles on each side of C O similar to the whole triangle A C D, and to each other. Now, $(AO \times OD) = OB^2$ or OC^2: but, Euclid nowhere proves this fact: nor could he, without travelling out of the domain of pure Geometry. It can, however, be demonstrated, by wielding "*that indispensable instrument of Science, Arithmetic.*" By analogy or proportion, $AO : OB :: OB : OD$, and, $AO : OC :: OC : OD$: but to find the value of O D, we must put a value on some line in the figure, and have recourse to "*that indispensable instrument of Science, Arithmetic.*" *

Now, by hypothesis, let $AO = 4$: then, $OB = 3$; by construction. Hence: $AO : OB :: OB : OD$; that is, $4 : 3 :: 3 : 2{\cdot}25$; therefore, $OD = 2{\cdot}25$; and it follows, that $(AO \times OD) = OB^2$; that is, $4 \times 2{\cdot}25 = 9 = OB^2$. It is self-evident, that B O D is a right-angled triangle; therefore, $BO^2 + OD^2 = BD^2$; that is, $3^2 + 2{\cdot}25^2 = 9 + 5{\cdot}0625 = 14{\cdot}0625 = BD^2$; therefore, $\sqrt{14{\cdot}0625} =$

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* See "Euclid's Elements of Plane Geometry." By W. B. Cooley, A. B. Appendix to the Second Book.

$3{\cdot}75 = BD$: and it follows, that $AB^2 + BD^2 = (AO + OD)^2$; that is, $5^2 + 3{\cdot}75^2) = (4 + 2{\cdot}25)^2$; or, $(25 + 14{\cdot}0625) = 6{\cdot}25^2 = 39{\cdot}0625 = AD^2$; therefore, $AD^2 = \sqrt{39{\cdot}0625} = 6{\cdot}25$. Now, A D is bisected at E; therefore, $ED = \frac{6{\cdot}25}{2} = 3{\cdot}125$. But, $ED - OD = EO$; that is, $3{\cdot}125 - 2{\cdot}25 = {\cdot}875 = EO$, and E O is the line that joins the middle points of the diagonals A D and C B in the quadrilateral A C D B. Hence: $\{AD^2 + CB^2 + 4\,(EO^2)\} = \{AC^2 + CD^2 + BD^2 + AB^2\}$; that is, $\{6{\cdot}25^2 + 6^2 + 4\,({\cdot}875^2)\} = \{5^2 + 3{\cdot}75^2 + 3{\cdot}75^2 + 5^2\}$; or, $\{39{\cdot}0625 + 36 + 3{\cdot}0625\} = \{25 + 14{\cdot}0625 + 14{\cdot}0625 + 25\}$, and this equation $= 78{\cdot}125$. But, $\{AC^2 + CD^2 + BD^2 + AB^2\} = 3\frac{1}{8}\,(AB^2)$; that is, $\{25 + 14{\cdot}0625 + 14{\cdot}0625 + 25\} = 3{\cdot}125 \times 25 = 78{\cdot}125$: and these equations = area of the circle X. Hence: 3·125 must be the true arithmetical value of π, which makes 8 circumferences = 25 diameters in every circle.

Now, to upset this Geometrical coach, Mathematicians must prove one of two things. They must either prove that π times the square of the radius is not equal to area in any circle; or they must prove that the sum of the squares of the four sides in a quadrilateral, is not equal to the sum of the squares of the diagonals in the same quadrilateral, together with four times the squares of the line that joins the middle points of the diagonals.

Now, my dear Sir, if you want to square the circle, or in other words, if you want to get a square exactly equal in superficial area to the circle X, I will show you how to find it. From the point G, draw a straight line—say G N—perpendicular to E D, making G N equal to G D. Produce G A to a point M, making G M equal to $\{2\,AG - GD\}$, and join M N. The square on M N will be the required square. (*I have indicated this square by dotted lines.*) For example: If $OA = 4$, then, $AG = 5$, and $GD = 1{\cdot}25$; therefore, $\{2\,AG - GD\} = \{10 - 1{\cdot}25\} = 8{\cdot}75 = MG$, and $GN = GD = 1{\cdot}25$; therefore, $MG^2 + GN^2 = 3\frac{1}{8}\,(AB^2)$; that is, $\{8{\cdot}75^2 + 1{\cdot}25^2\} = 3\frac{1}{8}\,(5^2)$; or, $\{76{\cdot}5625 + 1{\cdot}5625\} = 3{\cdot}125 \times 25$: and this equation $= 78{\cdot}125$ = area of the circle X, and area of the square on M N: and it follows, that the area of every circle is equal to the area of a square on the hypothenuse of a right-angled triangle, of which the sides that contain the right angle are in the ratio of 7 to 1, and the sum of these two sides equal to the diameter of the circle. In how many ways have I proved this fact by practical or constructive Geometry?

The discovery that Euclid is at Fault has opened up to me new methods of examining the geometry of a circle. The triangles on each side of O B in this interesting geometrical figure, are similar triangles, and similar to the whole triangle A B D; and it follows, that the triangles on each side of O C are similar triangles, and similar to the whole triangle A C D. Now, by Euclid: Prop 8: Book 6: it is made to appear that: *In a right-angled triangle, if a perpendicular be drawn from the right angle to the opposite side, the triangles on each side of it are similar to the whole triangle and to each other, and that this is of general and universal application.* This is not true! You will observe that, I have not said in "*Euclid at Fault*," that under no circumstances is it true, but that *it is not true under all circumstances.* The reason is this! Euclid in his fifth book on proportion; attempted to make his theorems of general and universal application, alike applicable to commensurables and incommensurables. In this, Euclid attempted an impossibility, and is of necessity at fault: that is to say, his theorems in the sixth book rest for their proofs on the fifth, and will not stand the test of "*that indispensable instrument of Science, Arithmetic*," and are therefore not true under all circumstances, and this may be demonstrated in many ways.

Well, then, Mathematicians *assuming* Euclid to be infallible, are led into the most extraordinary blunders. I will give you one instance. W. D. Cooley in his Appendix to the fifth and sixth books of Euclid, broadly asserts: "*To find two mean proportionals, or,* A *and* B *being given, to find* x *and* y, *so that* A : x : : y : B *is a problem beyond the reach of Plane Geometry.*" In a correspondence I am having with a gentleman, whose acquaintance I made at Norwich, I have proved the absurdity of this assertion in several ways: Mr. Cooley would be right, if π were *incommensurable*, and Euclid *not* at fault.

I was nearly forgetting what appears to me a very convincing proof of the true arithmetical value of π. Every Mathematician will admit that 2 π (radius) = circumference in every circle: and that circumference × semi-radius = area in every circle. Let us try what one of my correspondents calls "*the Seaforth mince π*" by this test. Then : $2\pi(AB) = 6{\cdot}25 \times 5 = 31{\cdot}25$ = circumference of the circle X : and, $\frac{2(AB)}{4} = \frac{10}{4} = 2{\cdot}5$ = semi-radius of the circle X : and it follows, that $(ED \times AB^2) = 2\pi(AB) \times \frac{2(AB)}{4}$, that is, $(31{\cdot}25 \times 2{\cdot}5) = (3125 \times 25)$, and this equation $= 78{\cdot}125$ = area of the circle X. I may put the following question to all opponents :—*What in the name of common sense can the arithmetical value of π be but* 3·125 ?

I know of no diagram so well adapted as the enclosed, to a Lecture such as yours. It is simple in construction, is based on the admitted properties of quadrilaterals, and the proofs by means of it, so far as "*Squaring the Circle*" is concerned, can be brought out simply and shortly, and even made perfectly intelligible to a mixed audience.

"*Euclid at Fault*" has brought me numerous correspondents, and kept me very busy while at Norwich. You would be greatly amused if you saw this correspondence, and I think it is very probable I may some day publish it. I had something more to tell you, with reference to "*Euclid at Fault*," and may write you again when I have a leisure day.

Believe me,
My dear Sir,
Very truly yours,
JAMES SMITH.

Now, my dear Sir, you cannot fail to be acquainted with Cooley's Elements of Plane Geometry ; and Cooley was a "*recognised Mathematician*." You either agree with him, or you do not, in the *assertion* I have quoted from the Appendix to his Fifth and Sixth Books. Perhaps you will be kind enough to say which alternative you adopt, and whether you would wish me to furnish the proof that Cooley is wrong.

December 1st.

I had written so far when I received your second Letter, under date 28th November, and this morning's post brought me your favour of yesterday. You admit, that if the longer sides in a parallelogram, of which the diagonals are 8 units of length, be 6·4, the shorter sides are 4·8, making area 30·72. And you also admit that the sides of an inscribed square, to a circle of diameter 8, are $4\sqrt{2}$, making area 32. These admissions I wanted, and nothing more, and am sorry to have given you more trouble than was necessary, for which I must apologise.

I am not very well to day, and must therefore conclude this epistle; but will take the earliest opportunity of replying to your last communication.

Believe me, my dear Sir,

Very truly yours,

JAMES SMITH.

THE REV. PROFESSOR WHITWORTH.

THE REV. PROFESSOR WHITWORTH *to* JAMES SMITH.

LIVERPOOL, *2nd December*, 1868.

MY DEAR SIR,

Your Letter has just reached me. You gave me what professed to be $\pi = \frac{25}{8}$. The whole cogency of your argument depended on the assertion contained in the words, "*we cannot get these equations by any other value of π.*"

I pointed out, that even such absurd suppositions as $\pi = \frac{16}{5}$ or $\pi = \frac{512}{125}$, or, if you like, the equally absurd $\pi = \frac{5}{2}$, or $\pi =$ any fraction, whose numerator and denominator are powers of 2 and 5 only, will, in your way of working, produce your vaunted equations, or (as Mathematicians would call them) identities. And you complain that I give no shadow of proof of the assertion that these values will thus apply. But the *onus probandi* is on your side. It is you that profess to prove that $\pi = \frac{25}{8}$. And an essential step of your argument is found in the statement, "*we cannot get these equations by any other value of π.*" Prove this statement, and your argument may become complete. Leave it unproved, and you are begging the whole question.

Now, Sir, may I ask what proof you have that no value of π, except $\frac{25}{8}$, will produce the said equation? You CANNOT prove it. But you have, perhaps, strong grounds for fancying it to be true, because you have (perhaps) tried one or two thousand other values for π which SEEMED to fail! In this case, the only ground you CAN take up in defence of your argument is to say, "*I assert that* $\pi = \frac{25}{8}$ *is the only value which will produce the identities, and I am ready to shew you that any other value you like to mention will fail.*" But it appears from your Letter this morning, that you are not prepared thus to maintain your assertion. We are expected to accept it without proof, simply on Mr. Smith's *ipse dixit;* and if we suggest two or three other values which will equally apply, nay more, if we suggest that some admittedly absurd values have just as much claim to acceptance as $\frac{25}{8}$ itself, they are thrown back at us, and the writer, who rests a whole argument on the *assertion* that nothing but $\frac{25}{8}$ will do, declines to consider or to try whether other absurd values will do or not.

Of course I do not want you to try the application of $\frac{16}{5}$ or $\frac{512}{125}$, to your argument, unless you like; but of course if you cannot defend the assertion that "*we cannot get these equations by any other value of* π" than $\frac{25}{8}$, your argument falls to the ground, and must be cast aside.

This is the only case, in which you have objected to the arguments, whereby I have shewn that each of your proffered proofs is defective.

You have given what you thought were proofs, and I have patiently, and I hope courteously, pointed out the fallacy of each one. We had better now rest until you are able to consider my observations, and to determine whether there still remains to you any one argument, which you still conceive to be sound.

But, I must point out a mistake you fall into respecting Mathematicians, and the ground they hold.

You speak of them as *assuming* Euclid to be true, and I perceive you continually introduce a theorem with such words as these:—"*The following was long ago discovered, and known to Geometers and Mathematicians.*" As if Mathematicians would accept a theorem without proof, because their predecessors accepted

t. Allow me to say that no Mathematician will accept an assertion as to a mathematical truth, either on the authority of Euclid or of any one else. We accept *nothing* unless we can PROVE it. We do not assume Euclid to be true, but we continually refer to results which Euclid gives, because we have followed out for ourselves the arguments by which he establishes them, and thus they are proved to us; but we do not expect any one else to accept them, unless he also for himself has followed out the reasoning by which they are established, and has satisfied himself of their truth.

I have hardly patience to read the long Letter, which you copy out for me, addressed by yourself to "D," because it is full of miserable personalities; but I see that you fall into your old fallacy of arguing, that because $\frac{25}{8}$ satisfies some conditions that π ought to satisfy, THEREFORE it is the true value of π; forgetting that is necessary to PROVE that $\frac{25}{8}$ is the ONLY value which satisfies the conditions. In all these so-called proofs you will arrive at the same result, if you try instead of $\frac{25}{8}$ any other fraction whose numerator and denominator are powers of 2 and 5 only. And the only reason why I have to specify such fractions as these is, that other fractions would either, in themselves or their reciprocals, generate decimals which would not terminate, and I find that you never work *accurately* with recurring decimals. You always take a few of the figures as "*sufficiently accurate for practical purposes*," and by the errors thus introduced, you arrive at discrepancies which you are unable to trace to their true origin.

I am very glad that the simple answer, which I gave to your two extraordinary questions of 28th November, contain "*the admissions*" which you wanted. Of course, all Mathematicians and all school boys, who have got as far as Euclid iii., would make the same "*admissions*" without the slightest hesitation. You are the only Geometer I ever met with who could have any doubt on the subject, and the only one who could speak of *the* dimensions of *the* inscribed rectangles in a given circle, as if they were limited in number, or as if there were one or two which could be regarded as THE inscribed rectangles, κατ' ἐξοχήν. (But now that you have the admissions, what will you do with them? W. A. WH.)

Your Letters have been very interesting, because they have laid open to me (what was before most marvellous) the process by which a person could work out for himself such a result as $\pi = \frac{25}{8}$, and be satisfied with it without professing to shew the fallacy of the reasoning of ordinary text books. You have evidently been led astray, by your habit of working decimals incorrectly. You have made calculations involving the value of the reciprocal of π, or involving division by π, and the decimals you have only carried to a finite number of places. Hence, discrepancies arose, which you found did not occur in working with $\pi = \frac{25}{8}$, because neither $\frac{1}{8}$ nor $\frac{1}{25}$ gives a recurring decimal. Hence, you too hastily concluded, that $\frac{25}{8}$ was the correct value of π. But if you would work with $\pi = 3{\cdot}14159$ or $\pi = \frac{22}{7}$, taking care to use vulgar fractions instead of recurring decimals, you would be unable to deduce a single inconsistency, or to maintain any ground for the preference of $\pi = \frac{25}{8}$.

You ask a question about a statement made by a Mr. Cooley, a man of whom I have never heard. If your inverted commas mark a true quotation from his work, of course he must be quite wrong. The problem (as you propose it) is not beyond the scope of pure Geometry, but is "*indeterminate;*" that is, it admits of an infinite number of solutions.

But possibly you quote from memory, and the explanation $A : x = y : B$ may be your own. If this be so, it seems probable that Mr. Cooley's definition of two mean proportionals is not what you take it to be. Thus, the problem may be to find two quantities, x and y, between A and B, so that $A : x = x : y = y : B$, or so that all the four are continued proportionals. Thus, the problem becomes a *determinate* one, of which the solution ($x = \sqrt[3]{A^2 B}$ and $y = \sqrt[3]{A B^2}$) can be written down at once algebraically, but is certainly beyond the reach of Plane Geometry, as far as I am a Geometrician.

Will you please say whether the quotation you have made is accurate, or whether my conjecture may be correct?

I am sorry to hear that you are unwell. Pray do not hurry yourself to reply to me.

Believe me, yours very truly,

JAMES SMITH, ESQ. W. ALLEN WHITWORTH.

P.S.—What about the Postcript to my last Letter?—W. A. WH.

BARKELEY HOUSE, SEAFORTH,
2nd December, 1868.

MY DEAR SIR,

Permit me to refer you to the diagram enclosed in my Letter of the 30th November *(see Diagram II.)*, posted yesterday, and ask you to favour me by drawing a straight line, joining B E. You will find, that the triangle B O E, constructed by this simple operation, is a right-angled triangle, of which the sides that contain the right angle are in the ratio of 7 to 24.* The triangle is a commensurable right-angled triangle, and when B O = 3, then, B E the hypothenuse = 3·125 = π. It is self-evident that B E = E D, for they are radii of the circle Y. You admit that in such a triangle the natural sine of the acute angle is ·28, but you doubt that the angle is an angle of 16° 16′, and ask me for the proof. I could give you the proof in many ways, but each would involve a diagram and long Letter, and I should have to travel over the same ground that I have already done with other correspondents. This is to me wearisome, and I think I can pursue another course, that will better serve your purpose, and be more convenient to me.

In the right-angled triangle A O B, the sides A O and O B, which contain the right angle, are in the ratio of 4 to 3, by construction: and when A O = 4, then, O B = 3, and A B = 5 : (A O + O B) = 7, and, 2 (A O × O B) = 24, and thus, we get the ratio between the sides that contain the right angle in the triangle B O E. Now, if we take any two consecutive numbers, and make their sum and twice their product the sides of a right-angled triangle, and contain the right angle, the triangle will be a commensurable right-angled triangle; and under all circumstances, the longer of the sides that contains the right angle will be less than the side that subtends the right angle by a constant quantity, represented by the arithmetical expressions 1, 1^2, or $\sqrt{1}$. Thus, from the numbers 1 and 2 we get the triangle of which the sides are 3, 4, and 5, and this I call the *primary* commensurable right-angled triangle, since it is the smallest triangle of which all the sides can be arithmetically expressed in whole numbers. From the numbers 2 and 3 we get the triangle of which the sides are 5, 12, and 13 : and from the numbers 3 and 4 we get the triangle of which the sides are 7, 24, and 25, and so on, *ad infinitum*.

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* The triangle B O E, and the triangle H P T in the Diagram in "*Euclid at Fault*," are similar right-angled triangles. Hence: the side P T in the triangle H P T = $\frac{7}{25}$ (H P), or, $\frac{7}{24}$ (B T), and it is self-evident that B T is a side of the right-angled triangle O B T.

In the triangle A O B the trigonometrical sine of the acute angle A is ·6 : and the trigonometrical sine of the obtuse angle B is ·8 ; and are in the same ratio as the sides that contain the right angle, when A O B represents the primary commensurable right-angled triangle.* In a Letter recently written to a very eminent Mathematician, I have dealt with a geometrical figure so constructed, as to contain three similar right-angled triangles, having the acute angle common to the three triangles : and another right-angled triangle quite dissimilar, but in which the side subtending the right angle is exactly equal to the side subtending the right angle in one of the three similar right-angled triangles, for they are radii of the same circle, by construction. By means of this geometrical figure, I have proved that if existing tables of natural sines and co-sines be correct, the acute angles in similar right-angled triangles may be arithmetically different. It is more than my health will permit to write Letters over and over again, or I might give you a copy of the Letter referred to. You are decidedly in error, if you mean to tell me that the natural sine and co-sine of an angle of 16° 16′, as given in tables —the one greater than ·28 and the other less than ·96—is not intended to convey the idea, that they are the true values of the natural sine and co-sine of the angle, as nearly as these values can be given to 7 places of decimals. Does not $1 - \cdot 9599684^2 = \cdot 2801083^2$ very nearly? Does not $1 - \cdot 96^2 = \cdot 28^2$ exactly? I have been told before, that ·28 is the natural sine of an angle of $16^\circ\ 16' - x$. It is not reasoning when Mathematicians assume tables to be infallible, and then boldly assert that I am wrong. Have I not a right to expect them to furnish the proof?

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* When A O B represents the primary commensurable right-angled triangle, the natural sine of the acute angle is 3, and the trigonometrical sine ·6. The natural sine of the obtuse angle is 4, and the trigonometrical sine ·8. Hutton gives the natural sine and co-sine of the acute angle as ·5999549 and ·8000338. But, by analogy or proportion, 3 : 4 : : ·5999549 : ·79993986 with 6 to infinity, and this is at variance with the ***known and indisputable*** ratio between the sides that contain the right angle in the triangle A O B. How then, can Hutton's Tables be correct?

Now, let A B C denote a right-angled triangle, B the right angle, and the sides A B and B C, which contain the right angle, be 10 units and 5 units in length. It cannot be disputed that such a triangle can be constructed. Then: $AB^2 + BC^2 = 10^2 + 5^2 = 100 + 25 = 125 = AC^2$; therefore, $\sqrt{125} = 11{\cdot}18034$ nearly $= AC$, the hypothenuse. $\frac{BC}{AC} = \frac{5}{11{\cdot}18034} = {\cdot}4472136$ is the arithmetical value of the natural (*trigonometrical*) sine of the acute angle: and, $\frac{AB}{AC} = \frac{10}{11{\cdot}18034} = {\cdot}8944272$ is the natural (*trigonometrical*) sine of the obtuse angle, and natural (*trigonometrical*) cosine of the acute angle. Hence: $\frac{4\,AC}{100} =$ the natural (*trigonometrical*) sine of the acute angle: $\frac{8\,AC}{100} =$ the natural (*trigonometrical*) sine of the obtuse angle: the sines of the acute and obtuse angles are in the ratio of 1 to 2: and the acute angle is an angle of 26° 34′, and answers to the angle F D E, in the diagram enclosed in my Letter of the 23rd November *(see Diagram I.)*, and is equal to half the angle F D C. You cannot fail to perceive that your argument—with reference to a right-angled triangle, of which the sides that contain the right angle are in the ratio of 2 to 1—falls to the ground; and I cannot help thinking, you will perceive that your reasoning is extremely fallacious.

I shall now direct your attention to some very remarkable facts that will be quite new to you. We know that, as the diameters of circles increase, the circumferences increase in *arithmetical* progression; but, we also know that, as the diameters of circles increase, the areas increase in *geometrical* progression. Hence: if we double the diameter of a circle we quadruple the area.

Now, referring to the diagram in my Letter of 30th November, I have proved that when $AO = 4$, then, $AD = 6{\cdot}25$. But, $\frac{AD}{2} = \frac{6{\cdot}25}{2} = 3{\cdot}125 = \pi =$ radius of the circle Y; and it follows, that $\pi^2 =$ area of the circle Y: that is, $(ED^2 \times \pi) = (3{\cdot}125^2 \times 3{\cdot}125) = 9{\cdot}765625 \times 3{\cdot}125 = 30{\cdot}517578125 =$ area of the circle Y. Now, by analogy or proportion, $ED : AB :: 3{\cdot}125 : 5$; and it follows, that the diameter of the circle Y, is to the diameter of the circle X, as 6·25 : 10. Hence, when E D = 3·125, then, A D : 2 A B : : 6·25 : 10. Now, the area of the circle Y = 30·517578125, and A D the diameter is to the area of the circle Y, as the diameter of the circle X is to $5\ (\pi^2)$: that is, 6·25 : 30·517578125 : : 10 : 48·828125. But, 10 times E D = 31·25, and the mean proportional between 31·25 and 48·828125 = $(2\ \pi)^2 = 6{\cdot}25^2 = 39{\cdot}0625$. Hence : the mean proportional between 10 E D and $5\ (\pi^2) = \sqrt{(31{\cdot}25 \times 48{\cdot}828125)} = \sqrt{1525{\cdot}87890625} =$ 39·0625, and is the area of a circumscribing square to the circle Y. Hence : the area of circle X is exactly equal to twice the area of a circumscribing square to the circle Y : that is, $3{\cdot}125\ (5^2) = 2\,(6{\cdot}25^2)$, and this equation = area of the circle X. You may invent as many values of π as you please, from fractions whose numerators and denominators are powers of 2 and 5, but by none of them could you produce these results, and I put the question : ***What, in the name of common sense, can the arithmetical value of π be, but $\frac{25}{8}$ = 3·125 ?***

Now, 3·125 (10) = 31·25 = circumference of the circle X : and, 8 (31·25) = 25 (10) = 250. Again : 3·125 (6·25) = 19·53125 = circumference of the circle Y : and, 8 (19·53125) = 25 (6·25) = 156·25 = 50 π. And it may be proved in a thousand ways, by practical or constructive Geometry, that 8 circumferences = 25 diameters in every circle. For the sake of argument, let it be granted, that I assume the *theory* which makes $\frac{25}{8}$ = 3·125 the value of π. Will you venture to tell me that Bacon was not in his senses when he penned the following remark ? "*Theoriarum vires, arcta et quasi se mutuo sustinente partium adaptatione, qua, quasi in orbem cohærent, firmantur.*" Am I to understand that, with "*recognised Mathematicians,*" Baconian philosophy is "*a mockery, delusion, and a snare ?*"

In the example of continued proportion, I gave you in my Letter of the 30th November, the product of the means is equal to the product of the extremes : that is, $A \times E = C^2 : A \times C = B^2 : B \times D = C^2$: and $C \times E = D^2$. From these facts it follows, that $2\ (\pi^2)$ is the mean proportional between $10\ (\pi)$ and $5\ (\pi^2)$: that is, $6{\cdot}25^2 = 39{\cdot}0625$ is the mean proportional between 31·25 and 48·828125 : and you cannot fail to perceive that we can work out the same result by means of the geometrical figure enclosed in that Letter.

From these facts we get a means of proof of the value of π, by the inscribed and circumscribed squares to a circle, of which I gave you an example in my Letter of the 23rd November, and which you have hardly condescended to notice. It will be convincing to any reflective Mathematician, and I shall give another example of it.

In the analogy or proportion, A : B : : B : C. When A denotes $\frac{3·125}{4}$ and B denotes 1, then, C = 1·28: that is, ·78125 : 1 : : 1 : 1·28, and the product of the means is equal to the product of the extremes. Hence: $\frac{·78125}{1}$ and $\frac{1}{1·28}$ are equivalent ratios, and it follows, that the product of any number multiplied by 1·28, is equal to the quotient of the same number divided by ·78125.

Now, let the area of an inscribed square to a circle be represented by the number 2. Then: $\{(2 + \frac{2}{4}) + \frac{1}{4}(2 + \frac{2}{4})\} = 3\frac{1}{8}(1^2)$, that is, $\{2·5 + ·625\} = (3·125 \times 1) = 3·125$ = area of the circle: and, $3·125\ (1·28) = 4$ = area of the circumscribing square to the circle. But, $\frac{3·125}{·78125} = 3·125 \times 1·28 = 4$ = area of the circumscribing square, and I presume you will not venture to tell me that the area of a circumscribing square to any circle is *not* the double of the area of an inscribed square. Well, then, will you be good enough to work out this result with the mysterious $\pi = 3·14159$, &c.? I need not shew *you* that, from a given area of a circle, we can find the areas of the inscribed and circumscribed squares; a thing impossible with $\pi = 3·14159$, or $\pi = 3·14159$ with any number of additional decimals. It can hardly be called fair reasoning to dispute my value of π, because we can work out certain results with $\pi = \frac{16}{5} = 3·2$, when you know as well as I do, that we can prove *mechanically* that π must be less than 3·2. According to my ethics, it would have been much fairer on your part to have at once frankly admitted that π is greater than 3 and less than 3·2.

3rd December.

I had written so far when your Letter of yesterday morning came into my hands. It has astonished me, and I am almost tempted to doubt, whether a desire to arrive at truth be your object. I shall wait your reply to this communication, and in the mean-time consider how I can best deal with your extraordinary epistle.

Believe me, my dear Sir,
Faithfully yours,
JAMES SMITH.

THE REV. PROFESSOR WHITWORTH.

P.S.—The quotation from Cooley is perfectly correct, and Cooley's Elements is a well known text book on Geometry.

THE REV. PROFESSOR WHITWORTH *to* JAMES SMITH.

QUEEN'S COLLEGE, LIVERPOOL,
4th, December, 1868.

MY DEAR SIR,

We do not "*assume Tables to be infallible*," and though it is quite true that the values given in Tables profess to be correct, as far as seven places of decimals go, yet no one supposes that the seven places express the true value of the sine or the Logarithm. When Mathematicians use such tables, they know that there may be an error; $= (\frac{1}{10})^7$ in the quotations from the Tables, and they always consider *to what degree* that error can affect their results. Your mistake is, that you think that if you use values true to seven places, they ought to produce results true to seven places, which they will not necessarily do. Thus, if you take the Logarithm of 3 from the Tables, true to seven places, and *thence calculate* the Logarithm of 3^{10}, it will only be obtained true to six places; and similarly in other cases. Values may be true, " as nearly as these values can be expressed in seven places of decimals," and yet the error may be quite sufficient to vitiate arguments which involve very small quantities.

You say, on page 5 of your Letter just received, that you will now direct my attention to some very remarkable facts, that will be quite new to me. And the first which you mention is certainly very remarkable and very new. You state it thus :—

"As the diameters of circles increase, the circumferences increase in arithmetical progression ; but, as the diameters increase, the areas increase in geometrical progression."

You do not say here by what law you suppose the diameters themselves to be increasing ; but, in order to make the first part of your statement true, we must suppose them to increase in arithmetical progression. We may, therefore, take a series of diameters of circles proportional to their numbers, 1, 2, 3, 4, 5, 6, etc., and then I quite agree with you that the circumferences will be in arithmetical progression, being also proportional to the numbers 1, 2, 3, 4, 5, 6, etc. But it is certainly very remarkable to be told that the areas will be in geometrical progression, commencing with 1, 4, and therefore proportional to the numbers 1, 4, 8, 16, 32, 64, etc., for I always thought the areas were proportional, as Euclid proves them to be (in Book vi.), to the squares of the diameters, and therefore to the numbers 1, 4, 9, 25, 36, 49, etc., which are not in geometrical progression at all.

My last Letter (which you had not received when you wrote your Letter of December 2nd), really answers nearly all your objections. And I do not think I can say anything more until I hear whether my former observations have satisfied or convinced you.

But I must say a word about the arguments by which you think you show some of my reasoning to be fallacious. You ask two questions :—

> "Does not $1 - ({\cdot}9599684)^2 = ({\cdot}2801083)^2$ very nearly?"
> "Does not $1 - ({\cdot}96)^2 = ({\cdot}28)^2$ exactly?"

I answer, to both questions, yes. And this proves that ·9599684 and ·2801083 are *very nearly* the co-sine and sine of some angle X; and that ·96 and ·28 are *exactly* the co-sine and sine of some other angle Y. You *assume* that Y = 16° 16′, which is not correct. The Tables (you say) tell us that X = 16° 16′ very nearly, or that X may be any angle, differing so little from 16° 16′, that its sine and co-sine are not affected to the seventh decimal place.

In your argument on pages 4 and 5, in which, as in your former Letter, you adopt a new definition of the term "obtuse angle," you say, $\sqrt{125} = 11{\cdot}18034$ *nearly;* but, in arguing from this statement, you drop the qualifying adverb *nearly*. I think you would find it convenient to use the symbol $\overset{\prime}{=}$ for "nearly equal," and add another accent for every step at which a new error is introdued. Thus :—

$$\sqrt{125} \overset{\prime}{=} 11{\cdot}18034.$$

And if you use this quantity as a divisor, and again neglect some figures of decimals, you would write *(e.g.)*—

$$\frac{17}{\sqrt{125}} \overset{\prime\prime}{=} 1{\cdot}52052.$$

Thus, we should understand exactly what you meant.

I grant that the result, "the area of the circle X is exactly equal to twice the area of a circumscribing square to the circle Y," could not be obtained by any value of π except that which you assign. But that only proves that your value of π and this result are both right or else both wrong. You assert they are both right; and I assert they are both wrong. But we want proofs, not assertions. We prove $\pi = 3{\cdot}14159\ldots\ldots$; you assert $\pi = 3{\cdot}125$. We ask you for a proof, and you say that no value of π, except yours, will make a certain circle X double of a certain square about Y. We ask why should this circle be double of this square, and you say because $\pi = 3{\cdot}125$, which is a *petitio principii.*

Let me observe, that it is only wasting time for you to send me new demonstrations, until you have either defended or abandoned your old ones.

But I would suggest a very practical test. Take a round table, five or six feet in diameter, and, with an *inelastic* tape, measure the diameter and circumference.

As you have a taste for Geometry, perhaps the following little question may interest you; I should be much obliged for a neat geometrical proof.

Let A, B, C, be the middle points of the sides B C, C A, A B, respectively, of any triangle ABC. And let AA, B B, C C, intersect in G. Also, let $A A_2$, $B B_2$, $C C_2$, be the perpendiculars from the angular points on the opposite sides (produced if necessary); and let G_2 be the point of intersection of $A A_2$, $B B_2$, $C C_2$. Also, let B, C, and $B_2 C_2$ intersect in P; C, A, and $C_2 A_2$ in Q; A, B, and $A_2 B_2$ in R; all the lines being produced, if necessary. Then prove that A P, B Q, and C R will necessarily be parallel to one another, and perpendicular to $G G_2$. Also shew that the triangle P Q R circumscribes the triangle A B C.

Believe me, my dear Sir,

Very truly yours,

JAMES SMITH, ESQ. W. ALLEN WHITWORTH.

P.S.—I am wondering whether you are going to adhere to your proof that a certain arc equals its own chord.

JAMES SMITH *to* THE REV. PROFESSOR WHITWORTH.

BARKELEY HOUSE, SEAFORTH,
4th December, 1868.

MY DEAR SIR,

After some consideration, I have arrived at the conclusion, that there is no necessity to wait your reply to my Letter posted yesterday ; and I shall pursue what appears to me the best course, with reference to our controversy, and proceed to test your capacity in constructive geometry.

Will you favour me by taking the diagram enclosed in my Letter of the 30th November *(see Diagram II.)*, and from the point E, the centre of the circle Y, draw a straight line parallel to A B, to meet a tangent of the circle Y, drawn from the point D, at a point—say X—producing a right-angled triangle E D X ? It will be self-evident to *you* that E D X and A O B will be similar right-angled triangles : and when the diameter of the circle $Y = 6{\cdot}25$, it follow of necessity, that $\frac{6\cdot25}{2} = 3{\cdot}125 = E\,D$, the radius of the circle Y. Now, these triangles have the sides that contain the right angle in the ratio of 4 to 3, by construction, and it follows, that $\frac{3}{4}(E\,D) = \frac{3 \times 3\cdot125}{4} = \frac{9\cdot375}{4} = 2{\cdot}34375 = D\,X$; therefore, $(E\,D^2 + D\,X^2) = (3{\cdot}125^2 + 2{\cdot}34375^2) = (9{\cdot}765625 + 5{\cdot}4931640625) = 15{\cdot}2587890625 = E\,X^2$.

Now, let A, B, C, D, and E denote the radii of circles, and let $A = 2 : B = 4 : C = 5 : D = 6{\cdot}25$: and $E = 7{\cdot}8125$.

If I were to ask you, how many times the area of the circle of which the radius is represented by A, is contained in the area of the circle, of which the radius is represented by E, I have no doubt you would solve the problem at once in the following way. Since the areas of circles are to each other as their radii, it follows, that $\frac{\pi E^2}{\pi A^2}$ solves the problem ; that is to say, π goes out, therefore, $\frac{E^2}{A^2} = \frac{7\cdot8125^2}{2^2} = \frac{61\cdot03515625}{4} = 15{\cdot}2587890625$ is the answer, and is equal to the area of a square on the hypothenuse of the triangle EDX. This I admit, and you will perceive that the area of the circle of which the radius is 2, is contained in the area of a circle of which the radius is 7·8125 as many times as there are units and parts of a unit in a square on the hypothenuse of a right-angled triangle, of which the sides that contain the right angle are in the ratio of 4 to 3, and the longer of these sides = 3·125.

Now, if A denote the area of the smallest of 5 circles, constructed with their radii in the proportions given, and be represented by any given number—say 60—it follows, that 60 (15·2587890625) = 915·52734375 will be the area of the circle represented by E. I need not tell *you* that, $\sqrt{15 \cdot 2587890625} = 3 \cdot 90625$, and I have shewn you the part that these figures play, when we assume $\pi = \frac{16}{5} = 3 \cdot 2$. (See my Letter of 30th November.)

May I ask you to construct a geometrical figure which shall contain 5 circles, of which the radii shall be in the proportions given; that is to say, of which the radii shall be 2, 4, 5, 6·25 and 7·8125 : and which shall also contain a right-angled triangle of which the sum of the areas of squares on the three sides of the triangle shall be 915·52734375, when the area of the smallest circle = 60?

You must not tell me the construction of such a geometrical figure is impossible; for, if you *can't* solve the problem, I *can*, and will solve it for you in due season, if you *don't*.

I have extracted from you certain admissions, with reference to which you *tauntingly* put the following question :—"*But now that you have the admissions, what will you do with them?*" I will relieve you from anxiety on this point, in my next Letter, if you do not divert me from my present intention.

Believe me, my dear Sir,

Faithfully yours.

JAMES SMITH.

THE REV. PROFESSOR WHITWORTH.

THE REV. PROFESSOR WHITWORTH *to* JAMES SMITH.

16, PERCY STREET, LIVERPOOL,
December 5th, 1868.

MY DEAR SIR,

I suppose I am to infer, from your attempts to change the subject, that you have nothing further to adduce in support of $\pi = 3{\cdot}125$; and that my strictures on your quasi-proofs are unanswerable. I have been waiting for any answer or objection you might bring against my demonstrations of the fallacies in all your proofs, and you say not a word on the subject; you do not even tell me whether you are going to maintain the equality of the chord and arc, which, *on your own shewing*, must exist, if $\pi = 3{\cdot}125$.

In your Letter just received, you propose "*to test my capacity for constructive geometry.*" And to this intent you propose a question, which, if it has one solution, has a million—like the last question which you propounded. You were "*satisfied*" with what you were pleased to call the "*admissions*" which I made on that question, which, by-the-bye, were "*admissions*" which no one would ever deny or dispute. But I still wonder what you will do with them, or how they will help you to prove $\pi = \frac{25}{8}$.

Do you not think it is rather inpertinent to our investigations, to send a question to test my capacities for constructive geometry? If I were one of those who made assertions, without proof, in a mathematical argument, resting conclusions not on argument but on my own character as a mathematician, it might be necessary to test my character and capacities. But I have not done so. My arguments stand or fall on their own merits; either they are conclusive (without a thought of the capacity of the writer), or they are fallacious. I have waited in vain for you either to shew that they are fallacious, or to admit that they are conclusive. But, instead, you persistently try to change the subject.

Now, if we are either of us to be the wiser for our correspondence, it can only be by following one subject till it is exhausted. The question at present is this:—Which is the true value of π? As soon as you have either proved $\pi = 3{\cdot}125$, or admitted that $\pi = 3{\cdot}141592...$, then I will answer your question about the constructive geometry, or consider any other subject you please. But my patient examination of all your proofs deserves some consideration. You ought not to be ashamed to say plainly that you cannot maintain those proofs any longer; or if this is not the case, you are surely bound to point out the fallacies of my strictures. I infer from your silence that you abandon the supposed arguments which I have shewn to be fallacious, but it would be more satisfactory to have it acknowledged by yourself.

I ask then, again, have I made any mistakes in proving that these proofs of yours are utterly unsound? If so, which are my mistakes? If not, have you any proof which you still think sound, that $\pi = 3{\cdot}125$? Are you going to allege the equality of a straight line and circular arc terminated by the same points?

Until these questions are answered, you have no right to expect me to enter on any extraneous discussions. Let us attend to one thing at a time. Either you can or you cannot prove $\pi = 3{\cdot}125$. Let us know which.

Believe me, my dear Sir,
Faithfully yours,
W. ALLEN WHITWORTH.

JAMES SMITH, ESQ.

JAMES SMITH *to* THE REV. PROFESSOR WHITWORTH.

BARKELEY HOUSE, SEAFORTH,
7th December, 1868.

MY DEAR SIR,

If you would write less, and think more, the sooner should we be likely to get to the end of our labours; but, if you pertinaciously persist in *assuming* that you have nothing to learn in Mathematics, and resolutely determine to take your stand in the ranks of that numerous class who "*despise wisdom and instruction*," I can't help it; and there is no telling how long our controversy may continue.

Your Letter of Saturday's date is before me, which you commence by observing:—"*I suppose I am to infer, from your attempt to change the subject, that you have nothing further to adduce in support of* $\pi = 3{\cdot}125$, *and that my strictures on your quasi-proofs are unanswerable.*" (Nonsense!) "*I have been waiting for any answer or objection you might bring against my demonstration of the fallacies in all your proofs, and you say not a word on the subject.*" (Simply untrue; if you had made an attempt to solve, and succeeded in finding the solution of the problem I gave you, in my Letter of the 4th instant, you would have made the discovery that the arithmetical value of π can be nothing else but $\frac{25}{8} = 3{\cdot}125$.) "*You do not even tell me whether you are going to maintain the equality of the chord and arc, which, on your own shewing, must exist, if* $\pi = 3{\cdot}125$." (Simply untrue. I answer this by putting a question:—Is not the natural sine of the acute angle, in a right-angled triangle of which the hypothenuse and shortest side are in the ratio of 8 to 1, as certainly ·125, as the natural sine of the acute angle, in a right-angled triangle of which the hypothenuse and shortest side are in the ratio of 2 to 1 is ·5?)

Paragraph 2. *In your Letter just received, you propose to test my capacity for constructive Geometry.*" (Why not?) "*And to this intent you propose a question which, if it has one solution, has a million, like the last question you propounded.*" (Simply untrue; and this I shall prove before I conclude this communication.) "*You were satisfied with what you were pleased to call the 'admissions' which I made on that question, which, by-the-bye, were admissions which no one would ever deny or dispute. But I still wonder what you will do with them, or how they will help you to prove* $\pi = \frac{25}{8}$." (I should have shewn you, in this communication, what I shall make of your admissions, had you not diverted me from the intention I expressed at the close of my Letter of the 4th instant. As it is, you must exercise a little patience, and wait my convenience, for this piece of information. I may, however, pay you the compliment of saying that you are about the first *profes-*

sional "recognised Mathematician," I have came in contact with, who has admitted anything. (The gentleman referred to in a former communication, as a "recognised Mathematician," and with reference to whom you tauntingly put the question to me, "recognised by whom ? ? ?" is *non-professional*, but is a "recognised Mathematician" by those who know him.) I only remember one exception to the rule: Professor de Morgan did admit that $\frac{\pi}{300}$ is the circular measure of an angle of 36 minutes, whatever be the value of π.

Paragraph 3. "*Do you not think it is rather impertinent to our investigations to send a question to test my capacity for constructive Geometry ?*" (Certainly not, if I think *that* the best method of convincing you of "*TRUTH.*") "*If I were one of those who made assertions without proof, in a mathematical argument, resting conclusions not on argument, but on my own character as a Mathematician, it might be necessary to test my character and capacities. But I have not done so.*" (Have you not given a method by which you say it is proved that $\pi = \frac{16}{\sqrt{3}} \left(\frac{1}{1 \cdot 3} + \frac{2}{5 \cdot 7} \cdot \frac{1}{9} +\right.$, &c.) *I say you have not proved this, nor can you, for it is not true;* and I may tell you that you have only yourself to thank for forcing upon me the necessity of testing your capacity for constructive Geometry. Will you venture to tell me that a circle has not the three properties of diameter, circumference, and area; just as certainly as a square has the three properties of side, perimeter and area? I trow not! Well, then, in due time, I shall give you a geometrical figure, in which there shall be a straight line exactly equal to the circumference of two circles in the same figure, and prove that whether the value of the diameter, circumference, or area of the circles, be the given quantity, we can find the values of the other two with arithmetical exactness.) '*My arguments stand or fall on their own merits, either they are conclusive (without a thought of the capacity of the writer) or they are fallacious.*" (True!) "*I have waited in vain for you either to show that they are fallacious, or to admit that they are conclusive.*" (I cannot feel assured that I should convince you that your arguments are fallacious and inconclusive; or that your first step in the search after π is based on an assumption that *never* has been, and never *can*, be proved; and were I to attempt to convince you on these matters, I fear our correspondence would be never-ending. I shall, therefore, adopt a different course.)

If we divide the circumference of a circle into any number of equal arcs, and from one of these arcs deduct $\frac{1}{25}$th part, the remainder multiplied by the number of arcs is a constant quantity, and is equal to the perimeter of a regular inscribed hexagon. For example: Let the circumference of a circle = 3·1416. Then: $\frac{3\cdot1416}{96} = \cdot032725$: and, $\frac{\cdot032725}{25} = \cdot001309$: therefore, ·032725 — ·001309 = ·03416 = $\frac{\pi}{100}$, according to Orthodoxy. But, 96 (·031416) = 3·015936, and this is the true arithmetical value of the perimeter of a regular inscribed hexagon to a circle of circumference = 3·1416; and is greater than the known arithmetical value of the perimeter of a regular inscribed hexagon to a circle of diameter unity. Again: Let the circumference of the circle = 3·125. Then: $\frac{3\cdot125}{25} = \cdot125$: $\frac{\cdot125}{25} = \cdot005$: therefore, ·125 — ·005 = ·12. But, 25 (·12) = 3, according to Heterodoxy; and yet, 3 is the known and indisputable arithmetical value of the perimeter of a regular inscribed hexagon to a circle of diameter unity. Again: Let the circumference of a circle = 3·125. Now, *we know* that the perimeter of a regular inscribed hexagon to a circle of radius 1 = 6: and, *we know* that the area of a circle of radius 1, and the circumference of a circle of diameter unity, are represented by the same arithmetical symbols, whatever be the value of π. Well, then, $\frac{3\cdot125}{6} = \cdot52083333$, with 3 to infinity: $\frac{\cdot52083333}{25} = \cdot02083333$, with 3 to infinity: therefore, ·52083333 — ·02083333 = ·5. But, 6 (·5) = 3, the known and indisputable value of the perimeter of a regular inscribed hexagon to a circle of diameter unity. *What, then, "in the name of common sense," can the arithmetical value of π be, but* $\frac{25}{8} = 3\cdot125$?

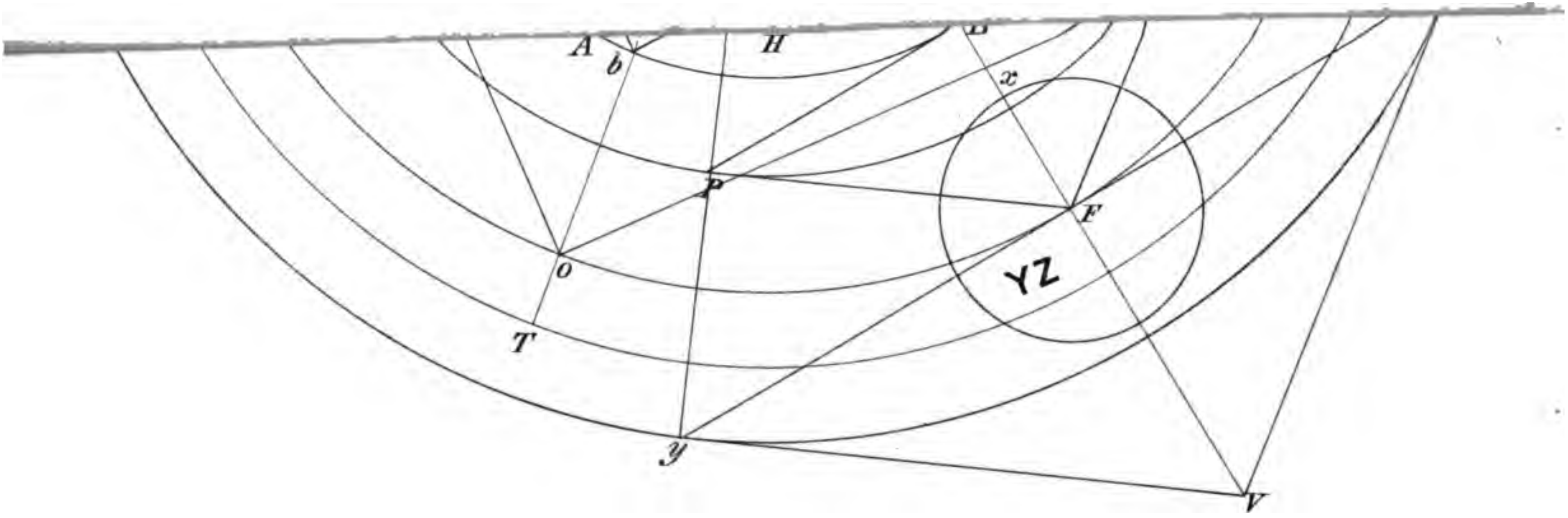
Now, my dear Sir, you may call this either a mathematical or a geometrical coach (for, "a rose by any other name would smell as sweet"); but, I defy you to upset the coach. You may divide the circumference into 7, 9, 21, 27, or any other number of equal parts you please, the result will be the same.

In the enclosed geometrical figure, *(See Diagram III.)* the equilateral triangle O A B is the generating figure of the diagram. The angle A O B and its opposite side A B are bisected by the line O H, by construction. Are not the angles H O A and H O B similar angles of 30° ? Is not O A to A H, or, O B to B H, in the ratio of 2 to 1 ? Are not the natural sines of the acute angles in the right-angled triangles O H A and O H B $= \frac{1}{2} = \cdot 5$? These facts I dare you to dispute ! Well, then, can I not charge you, with equal propriety, with making the chord A B $\neq$ the arc A B, as you can charge me, with making the chord and arc equal, in a bisected isosceles triangle which produces two right-angled triangles, in which the hypothenuse and shortest side are trigonometrically in the ratio of 8 to 1 ?

Now, let O K the radius of the circle P $= 2$. Then : O B the radius of the circle X $= 4$: O C the radius of the circle Y $= 5$: O F the radius of the circle M $= 6 \cdot 25$: and, O R the radius of the circle XZ $= 7 \cdot 8125$: by construction. The triangles O B C, O C F, O F R, and O R V are similar right-angled triangles, and have the sides that contain the right angle in the ratio of 4 to 3, by construction. There is no similar right-angled triangle within the equilateral triangle O A B. But, if we draw a straight line joining A K, then, this line will intersect the line O P at a point—say X—and since A K will be parallel to P C, it follows, that O K X will be a similar right-angled triangle to O B C, O C F, &c. : that is to say, will be a right-angled triangle, of which the sides that contain the right angle are in the ratio of 4 to 3. But, O P = O C, for, they are radii of the circle Y : and O *o* = O F, for, they are radii of the circle M : and, $2 (O F^2) = 3\frac{1}{8} (O C^2)$ or, $3\frac{1}{8} (O P^2)$ = area of the square *n o p m*, standing on the circle Y ; and it follows, that the square *n o p m* and the circle Y are exactly equal in superficial area, and makes 8 circumferences = 25 diameters in every circle, making $\frac{25}{8} = 3 \cdot 125$ the true arithmetical value of π.

Now, my good Sir, the "*onus probandi*" rests with you to controvert these facts, and relieves me from the necessity of controverting what you are pleased to term the fallacies of my reasoning. The fact is, your "*strictures*" are a mere exhibition of the "*power of assertion, not the force of reasoning.*"

DIAGRAM III.



You cannot expect me to wade through all the properties of this remarkable geometrical figure; if you are, what you profess to be, an authority in constructive geometry, you can trace out these properties without my assistance. It has more than once been said to me, "*Stick to Algebra;*" and *you*, in one of your communications, speak of working algebraically, as if Algebra could be made to override "*that indispensable instrument of science, Arithmetic,*" upon which it is founded.

In my Letter of the 4th December, I have shewn that the area of the circle P is contained 15·2587890625 times in the area of the circle X Z. Hence: when the area of the circle P = 60, then, 60(15·2587890625) = 915·52734375 = area of the circle X Z. Now, the sum of the squares of the three sides of the right-angled triangle O R V = $3\frac{1}{8}$ (O R^2), and is equal to the area of the circle X Z. Proof: Since πr^2 = area in every circle, it follows, that $\sqrt{\frac{\text{area}}{\pi}}$ = radius in every circle. Now, $\sqrt{\left(\frac{915·52734375}{3·125}\right)} = \sqrt{(292·96875)}$ = O R, $\frac{3}{4}$(O R) = $\sqrt{(164·794921875)}$ = R V, and $\frac{5}{4}$ (O R) = $\sqrt{(457·763671875)}$ = O V, therefore, O R^2 + R V^2 + O V^2 = (292·96875 + 164·794921875 + 457·763671875) = 915·52734375 = $3\frac{1}{8}$(O R^2) = area of the circle X Z.

I have no doubt *you* would have very little hesitation in telling me that to produce this result, I have assumed the value of π: but I may tell you, that if your object be TRUTH, you will not overlook the proof that precedes it, that $\pi = \frac{25}{8} = 3·125$.

Now, if with O as centre and O V as radius we describe another circle, and from the point V draw a straight line at right angles to O V, and therefore tangental to the circle, to meet O R produced at a point—say X—and so, constructing another right-angled triangle O V X. This triangle will be similar to the right-angled triangles O B C, O C F, O F R, and O R V: that is to say, the sides that contain the right angle will be in the ratio of 4 to 3; and when the radius of the circle P = 2, O V the radius of the circle O V X = π^2 = 9·765625. Now, we know the formula for finding the number of times the area of the circle P is contained in the area of the circle X Z: and by the same formula, the area of the circle P is contained in a circle of which O V is the radius $23{\cdot}841857910 15625$ times. Well, then, let the area of the circle $P = 60$. Will you work out the calculations, and find the arithmetical values of the sides of the triangle O V X? I think the results will surprise you. The area of a circle of which the diameter is 5, is $3\frac{1}{8}\ (2{\cdot}5^2) = 3{\cdot}125 \times 6{\cdot}25, = 19{\cdot}53125$, and this is the value of the perimeter of the right-angled triangle O V X, when the radius of the circle $P = 2$.

In your Letter of the 16th November, you said:—"*My object is not contention, my only desire being that truth should prevail. Any argument must cease to be edifying as soon as either party seeks for victory rather than for the TRUTH.*" How do you reconcile this language with your last Letter? It is impossible to read that communication without being led to the conclusion, that you have made up your mind to the resolute determination to write me down; but, I venture to assure you that you will certainly fail. I would advise you to be a little more careful for your own sake, and not attempt to play too desperate a game.

Believe me,
My dear Sir,
Faithfully yours,
JAMES SMITH.

THE REV. PROFESSOR WHITWORTH.

THE REV. PROFESSOR WHITWORTH *to* JAMES SMITH.

LIVERPOOL, *December 8th*, 1868.

MY DEAR SIR,

The threats with which you solemnly close your Letter, are even more amusing than the tone of lofty superiority in which your opening paragraph is written; but I have not time to waste on personalities.

I am afraid your jokes are rather lost upon me, as I am very obtuse in such things. I cannot see the joke of calling a mathematical proof a "*coach;*" nor can I perceive the connexion between the "*coach*" and the quotation from Shakspeare.

I am quite willing to admit, in reply to one of your questions, that a circle has diameter, circumference, area: and that a square has sides, diagonals, perimeter, area, and as many more "properties" as you please. But I cannot see how I ever seemed to deny any of these facts.

I will come to all your arguments in time. But I have not yet done with the proof you gave me, that a certain chord was equal to its arc. I have asked you several times, whether you still maintain that they are equal, and you decline to give an explicit answer. In your Letter to-day, you say—

"I answer this by putting a question. Is not the natural sine, &c." (To this question, I answer *yes*, and the admission is at your service). But this question gives no answer to my question, and before we go any further, I must ask you to tell me, *as explicity as possible*, whether you still maintain the equality of chord and arc.

To the other questions you have given me, I reply, in your own fondness for wandering beyond the mother tongue, ὅτι οὐσία πρὸς γένεσιν, ἐπιστήμη πρὸς πίστιν καὶ διάνοια πρὸς εἰκασίαν ἐστι.

I thank you for your definition of the term "Recognised Mathematician." I think it is very neat. I really believe, that if you could furnish a list of definitions of all the terms you use, there would be little to dispute between us.

I think "*nonsense*" and "*simply untrue*," are the concisest possible answers that can be given to a correspondent's arguments, and I congratulate you on having devised such a simple way of setting me aside.

If it were quite courteous, I would conclude with the paragraph with which your Letter opens.

Believe me,
My dear Sir,
Yours very faithfully,

JAMES SMITH, Esq. W. ALLEN WHITWORTH.

Now, Sir, you will observe that in the second paragraph of my Letter of the 7th Dec. to the Rev. Professor Whitworth, I put the following question to him: "*Is not the natural sine of the acute angle, in a right-angled triangle of which the hypothenuse and shortest side are in the ratio of* 8 *to* 1, *as certainly* ·125, *as the natural sine of the acute angle, in a right-angled triangle of which the hypothenuse and shortest side are in the ratio of* 2 *to* 1, *is* ·5?" You will not fail to perceive that the Professor gives the following reply: "*To this question I answer,* YES, *and the admission is at your service.*" If the Reverend and learned Professor had been a little more reflective, and a little less impulsive, he would have seen that this admission "*upsets*" his Mathematical "*coach.*" It appears to me that the *profundity* of his reflection is on a par with the *obtundity of his intellect**—as to the appreciation of a joke, or the distinction between the natural and trigonometrical sine of an angle.

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* His Grace the Duke of Devonshire—then Earl of Burlington—was President of "The British Association for the Advancement of Science," 1837-1838. At the inaugural meeting, held in the Theatre Royal, Liverpool, a gentleman of great wealth and high position in that town, moved or seconded a resolution,—my memory does not serve me to say which—but he talked such an amount of *nonsense*, that some of his friends on the platform stepped forward, and pulled his coat tails, by way of hinting to him that he had better bring his speech to a close; but *apparently* there were no means of stopping him. At first there was a titter, then a laugh, then a still louder laugh, until not a word he said could be heard, when an Irish gentleman sitting beside me observed:—"*Did you ever see such obtundity of intellect?*" At length, the Earl of Burlington rose, and lifting up his hands, exclaimed: Order! Order! Order! Instantly there was a dead silence; when the noble Earl observed:—*I hope you will give a patient hearing to so distinguished a Member of the British Association for the Advancement of Science.* The Liverpool philosopher was unable to finish his speech; and did not utter another word. His Grace the Duke of Devonshire is still in the land of the living, and if this anecdote should meet his eye, I have no doubt that his memory will serve him to vouch for the major part of its accuracy.

It may be admitted that in certain angles—such as angles of 45° and 30°—the natural and trigonometrical sines are *arithmetically* the same, to a circle of radius 1. But, are they not *arithmetically* the same in all angles, according to the *logic* of Mathematicians? You, Sir, can answer this question, and I may tell you that the blunderings into which Mathematicians have been led in their application of Mathematics to Geometry, is involved in the answer to this question. It is the *fallacious assumption*, that the natural sine and trigonometrical sine are the same in all angles, that leads Mathematicians to treat the trigonometrical functions of angles as lengths, and not as ratios of one length to another — *whether the assumption be made wittingly or unwittingly matters not, so far as regards the fact*—and this is the fallacy into which the Reverend Professor Whitworth has fallen, in the postscript to his second Letter of the 28th November.

Referring to the geometrical figure in connection with Professor Whitworth's postscript (page 38), he makes A B and A C = 1, and A B C denote one of 25 equal isosceles triangles inscribed in a circle; and he has given an affirmative answer to the question I put to him in the second paragraph of my Letter of the 7th December. Now, I would ask you, Sir, what especial charm is there about a circle of radius unity? I know of none!! If A B and A C = 1, then, the ratio of AB to BD is *(trigonometrically)* as 8 to 1. If A B and A C = 20, then, the ratio of A B to B D is *(trigonometrically)* as 20 to 2·5, or as 8 to 1. If AB and A C = 60, then, the ratio of AB to BD is *(trigonometrically)* as 60 to 7·5, or as 8 to 1. And, if the circumference of the circle = 360, then, the ratio of AB to B D is *(trigonometrically)* as 57·6 to 7·2, or as 8 to 1. Well, then, the angle B A C is an angle of 14° 24′, and the angles D A B and D A C are angles of 7° 12′, and the angles (expressed in degrees) are invariable, whatever arithmetical value we may put upon the radius of the circle. Hence, we may give as many arithmetical values as we please to the line B D, by changing the value of A B and B C, which are radii of the circle, but we cannot make the trigonometrical ratio of A B to B D other than 8 to 1, or the trigonometrical sine of the angles D A B and D A C other than $\frac{1}{8}$ = ·125.

If A B C denote one of 24 isosceles triangles inscribed in a circle, the angle B A C will be an angle at the centre of the circle, of 15°, and subtended by an arc of 15°. Hence: $\frac{24}{25}$ (15°) = $\frac{24 \times 15^\circ}{25}$ = 14° 24′ = the chord B C subtending the angle B A C, and the ratio of chord to arc is as 14° 24′ to 15°, and this is the invariable ratio between the sides of a regular polygon inscribed in a circle and their subtending arcs, whatever be the number of the sides of the inscribed polygon. It is self-evident, that as we increase the number of sides of a polygon inscribed in a circle, the sides vary both *geometrically* and *arithmetically* at every step, but not so the ratio between the sides and their subtending arcs!! Now, the angle B A C is bisected by the line A D, and it follows, that DAB and DAC are angles of 7° 30′, and the ratio of chord to arc is unchanged, whether A B C denotes one of 24 or one of 25 isosceles triangles inscribed in a circle. Hence: 7° 30′ : 7° 12′ : : 15° : 14° 24′, and it follows, that an arc has its sine as certainly as an angle, and *geometrically* the sine of an arc is half the chord of twice the arc : but, *trigonometrically* and *arithmetically* the sine of the arc varies, as we increase the number of the sides of a polygon inscribed in a circle.

BARKELEY HOUSE, SEAFORTH,
8th December, 1868.
Posted 10*th.*

MY DEAR SIR,

When I get your reply to my Letter of yesterday, I shall discover whether the proofs I have given you in that communication, that $\pi = \frac{25}{8} = 3{\cdot}125$, are satisfactory ; if not, I suppose I must adduce some other proofs in support of $\pi = 3{\cdot}125$. In the meantime I may be preparing to furnish you with the information you appear so anxious to obtain, with reference to the *admissions* you have made.

If any number be multiplied by itself, the product is a square number, the number 1 excepted, which cannot be made *arithmetically* greater by multiplying it by itself, or *arithmetically* less by extracting its root. If a decimal quantity be multiplied by itself, the product is *arithmetically* less than that quantity, and if we extract the root of a decimal quantity, it is *arithmetically* greater than that quantity. Thus, $\cdot25^2 = \cdot0625$, and is less than $\cdot25$; and $\sqrt{\cdot96}$ is $\cdot979$, &c., and is greater than $\cdot96$.

Now, the *mystic* number 4 is a square number, being the product of 2 multiplied by 2. Hence : $(\sqrt{4} + \sqrt{4})$ and $(2 + 2)$ are equivalent arithmetical expressions, or identities, and are equal to $\sqrt{4 \times 4} = \sqrt{16} = 4$; but, in this case, we may first extract the roots, and the sum of the roots gives the true result. Let the side of a square be represented by $\sqrt{4}$. Then : $(\sqrt{4^2} + \sqrt{4^2}) = \sqrt{4 + 4}$ $= \sqrt{8}$ is the arithmetical value of the diagonal of the square.

Now, let the side of a square be represented by $\sqrt{8}$. Then: $(\sqrt{8^2} + \sqrt{8^2}) = \sqrt{8 + 8} = \sqrt{16} = 4 =$ diagonal of the square. But, if we first extract the root of a side of the square, and double it, we get a very different result. $\sqrt{8} = 2{\cdot}8284$, &c., and $2\ (2{\cdot}8284) = 5{\cdot}6568$, and is a quantity far in excess of the true arithmetical value of the diagonal of the square. There is more in these facts than strikes the eye, and they have never received the attention they demand at the hands of Mathematicians. Their importance becomes obvious in dealing with practical or constructive Geometry.

Permit me to refer you to the diagram in "*Euclid at Fault.*" K B and L H are diameters of the circle; O B and O H are radii of the circle; and O B is divided into four equal parts, by construction; and it follows, that K F = 5, and F B = 3, when K B = 8.

Now, according to Euclid, the square of a line drawn from the circumference of a circle perpendicular to its diameter, is equal to the rectangle under the segments of the diameter. Euclid nowhere proves this, nor could he, by pure Geometry; but, *apparently*, it is very readily demonstrated by applied Mathematics. For example: When H O, a radius of the circle, = 4, then, O F = 1. Hence: $(HO^2 - OF^2) = (KF \times FB)$, that is, $(4^2 - 1^2) = (5 \times 3)$, or, $(16 - 1) = (5 \times 3) = 15$, and this equation, or identity—if you like the term better—is *apparently* equal to HF^2.

For what I am about to bring under your notice, I shall assume Euclid not to be at fault, and I think you will hardly find fault with me on this account.

By Euclid: Prop. 47: Book 1:—

$HF^2 + KF^2 = KH^2$; that is, $(\sqrt{15^2} + 5^2) = (15 + 25) = 40 = KH^2$, therefore, $KH = \sqrt{40}$, when the diameter of the circle $= 8$.

By Euclid: Prop. 12: Book 2.

$HO^2 + OK^2 + 2(OK \times OF) = KH^2$, that is, $\{4^2 + 4^2 + 2(4 \times 1)\} = KH^2$, or, $(16 + 16 + 8) = 40 = KH^2$, therefore, $KH = \sqrt{40}$.

By Euclid: Prop. 8: Book 6.

$KH^2 = KB \times KF$, therefore, $\sqrt{8 \times 5} = \sqrt{40} = KH$.

On all these "*shewings*" $KH = \sqrt{40}$.

I shall now proceed to prove, that $\sqrt{15}$ is not universally $\sqrt{15}$, when applied to practical or constructive geometry. Now, according to Euclid, $HF = \sqrt{15}$ when the diameter of the circle $= 8$, and HFB is a right-angled triangle; therefore, $HF^2 + FB^2 = (\sqrt{15}^2 + 3^2) = (15 + 9) = 24 = HB^2$; therefore, $HB = \sqrt{24}$. But, $PB = HF$, for they are opposite sides of the parallelogram $FBPH$, of which HB is a diagonal.

Now, by analogy or proportion, $KH : HB :: HB : HM$, that is, $\sqrt{40} : \sqrt{24} :: \sqrt{24} : \sqrt{14 \cdot 4}$; therefore, $HM = \sqrt{14 \cdot 4}$. But $(HB^2 + HM^2) = BM^2$; that is, $(\sqrt{24}^2 + \sqrt{14 \cdot 4}^2) = (24 + 14 \cdot 4) = 38 \cdot 4 = BM^2$; therefore, $\sqrt{38 \cdot 4} = BM$. But, by analogy or proportion, $KF : FH :: KB : BM$, that is, $5 : \sqrt{15} :: 8 : \sqrt{38 \cdot 4}$, again making $BM = \sqrt{38 \cdot 4}$. Again: By analogy or proportion, $KF : FH :: HP : PM$; that is, $5 : \sqrt{15} :: 3 : \sqrt{5 \cdot 4}$; therefore, $PM = \sqrt{5 \cdot 4}$. But, $(HM^2 - HP^2) = (\sqrt{14 \cdot 4}^2 - 3^2) = (14 \cdot 4 - 9) = 5 \cdot 4 = PM^2$, therefore, $PM = \sqrt{5 \cdot 4}$; and again we prove in two ways that $PM = \sqrt{5 \cdot 4}$. Well, then, I am sure you will admit that $BM = \sqrt{38 \cdot 4}$: and, that $PM = \sqrt{5 \cdot 4}$, when the diameter of the circle $= 8$. Now, $MB - PM = PB$, and $PB = HF = \sqrt{15} = 3 \cdot 872$ &c., and although $\sqrt{15}$ is a definite arithmetical expression, it is nevertheless an irrational quantity, so that if we extract the root to 100 places of decimals, there would still be a remainder, and consequently the figures so obtained would not truly represent the length of the line H F.

Now, the sides that contain the right angle in the right-angled triangle O B T are in the ratio of 4 to 3, by construction. This makes the sides that contain the right angle in the triangle H P T in the ratio of 24 to 7, by construction. Hence: H P T is a similar triangle to the triangles D G H and A F M in the geometrical figure represented by the diagram enclosed in my Letter of the 23rd November. (*See Diagram I.*) Now, when the diameter of the circle $= 8$, H P the longest of the sides that contain the right angle in the triangle $HPT = 3$, and makes $PT = \frac{7}{24}(HP)$ or, $\frac{7}{24}(BT) = \cdot 875$. And, $(HP^2 + PT^2) = (3^2 + \cdot 875^2) = (9 + \cdot 765625) =$

9·765625 = H T^2; therefore, $\sqrt{9·765625}$ = 3·125 = H T = π. Hence: H P is to H T as the perimeter of a regular hexagon to the circumference of its circumscribing circle, when the diameter of the circle is represented by unity: and it follows, that when $\sqrt{15}$ represents the length of a line, it *really* represents the *arithmetical* quantity 3·875, although when we extract the root we can only make it 3·872, &c. Pray take these facts in connection with my Letter of the 23rd November!!

Well, then, F B P H is a rectangular parallelogram, and is divided by H B, the diagonal, into two similar and equal right-angled triangles. Take the triangle H B P. Then: H T B, a part of it, is an oblique-angled triangle; therefore, according to Euclid: Prop. 12: Book 2: $\{HT^2 + TB^2 + 2(TB \times TP)\} = HB^2$; that is, $\{3·125^2 + 3^2 + 2(3 + ·875)\}$; or, $\{9·765625 + 9 + 5·25\}$ = 24·015625 = H B^2. Hence: $\frac{24}{25}(OT^2) + \frac{1}{7}(TP)^2$; = $\frac{24}{25}(5^2) + ·125^2$); that is, $(24 + ·125^2) = (24 + ·015625)$ = 24·015625 = H B^2: or, in other words, H B^2 is greater than $\frac{24}{25}(OT^2)$ by $\frac{\pi}{200}$. Will you be good enough to find the arithmetical values of H T and P T, and prove by Euclid: Prop. 12: Book 2: that $HT^2 + TB^2 + 2(TB \times TP) = HB^2$?

Now, the acute angle O, in the triangle O B T, is an angle of 36° 52′, and the obtuse angle an angle of 53° 8′: the acute angle H, in the triangle H P T, is an angle of 16° 16′, and the obtuse angle an angle of 73° 44′. Hence: the acute angles of the two triangles are together equal to the obtuse angle of the triangle O B T; and the four angles B O T, O T B, T H P and H T P are together equal to two right angles.

Is not Euclid at fault in making his book on proportion alike applicable to rational and irrational quantities? If not, are not Mathematicians at fault in their applications of Mathematics to Geometry? I think both are wrong, and I have given you my reasons for thinking so. The "*onus probandi*" rests with you to controvert them. *One thing is certain, Euclid and Mathematicians cannot both be right!!*

9th December.

I had written so far when your Letter of yesterday came to hand. You say :—"*Before we go any further, I must ask you to tell me, as explicitly as possible, whether you still maintain the equality of chord and arc.*" When, or where, have I ever said anything so absurd ? It is an easy matter to catch at a statement and pervert it, when it is very difficult to controvert it by argument. I not only answered this question by putting another, but I answered it in another way. Have you forgotten that the trigonometrical functions of angles are not lengths, but ratios of one length to another ? If you inscribe 25 equal isosceles triangles within a circle, will not the angles at the centre be angles of 14° 24′ ? Do we not, for all practical purposes, divide the circumference of the circle into 360° ? Well, then, if we take one of these triangles, and bisect the angle at the apex and its opposite side, dividing it into two similar and equal right-angled triangles, the trigonometrical ratio between the hypothenuse and shortest side in these triangles is as 8 to 1, and $\frac{1}{8} = \cdot 125$ is the trigonometrical sine of the acute angles. If the hypothenuse of the triangles be 57·6, and the base of the isosceles triangle 14·4, the ratio of chord to arc is as 14·4 to 15. The "*onus probandi*" rests with you to controvert this fact. How many things have I already proved ? Where have you ever attempted to controvert any of them ? Try a 24-sided polygon, of perimeter 360, inscribed in a circle, by the rule I gave you in my last Letter, and you will get at the ratio of chord to arc.

If there was anything discorteous in my last Letter, you may thank yourself for it. It was the dogmatical and dictatorial tone of yours that provoked it. I may tell *you*, that I had a correspondence which extended over a period of 18 months, with a gentleman about my own age, a St. John's College man, and a Clergyman, and although we agreed to differ at last, he admitted that nothing had escaped me in the course of our correspondence, "*unbefitting a Christian and a gentleman.*" I can assure you, if you play a fair game with me, you shall have nothing to complain of in that respect.

Believe me, my dear Sir
Faithfully yours,

THE REV. PROFESSOR WHITWORTH. JAMES SMITH.

P.S.—The last two pages are slovenly written, but I think you will understand my meaning. I am very unwell to-day, and could not manage to re-write them.

THE REV. PROFESSOR WHITWORTH *to* JAMES SMITH.

LIVERPOOL, *December* 11*th*, 1868.

MY DEAR SIR,

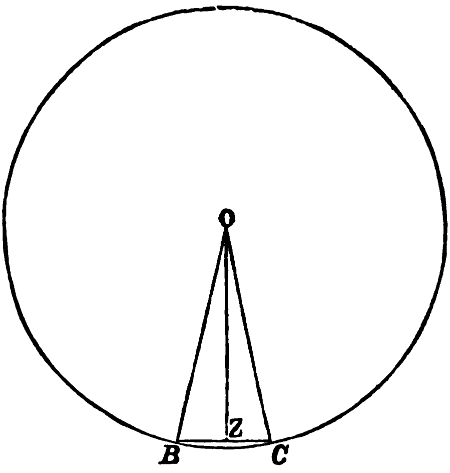
"If you refer to that 'Postscript' in which I pointed out that you make a certain arc and chord equal, you will observe that I say nothing about trignometrical functions, and therefore you cannot answer my objection by asking, have I forgotten that the trignometrical functions are not lengths but ratios? And you are not to think that I assumed a circumference to be equal to 360, for any such absurd reason, as that four right-angled triangles are often divided into 360 degrees. I simply accepted your calculations, and pointed out, that on your own shewing your arc and chord are equal."

"When you say:—By hypothesis, let the circumference of the circle = 360, you can only be understood to mean, let us take $\frac{1}{360}$th part of this circumference as unit of length. You may refer your measurements to any unit of length you please, but you must stick to the same unit for all your measurements or your comparisons fail."

"Your inconsistency is shown quite distinctly (I think), as follows; but I have yet to learn, in what point my 'Postscript' is unsatisfactory to you, as the objection about trignometrical ratios does not apply."

"Let A B C D......be successive angular points of a regular xxv.-gon inscribed in a circle, and let fall a perpendicular on the side B C."

"Then, by Mr. Smith's Letter of November 23rd, folio 10: * $\frac{BZ}{BO} = \frac{7 \cdot 2}{57 \cdot 6} = \cdot 125 = \frac{1}{8}$, or, B Z is $\frac{1}{8}$th of BO. And, therefore, B C is the $\frac{1}{4}$th of B O, and therefore the perimeter of the polygon is $\frac{25}{4}$ of B O, or, $\frac{25}{8}$ of the diameter."



"But, according to Mr. Smith's value of π, the whole circumference of the circle is $\frac{25}{8}$ of the diameter. Therefore, on Mr. Smith's hypothesis: Perimeter of inscribed polygon = circumference of circle; therefore, chord B C = arc B C."

"You will perceive that there is no hypothesis of mine involved in this reasoning; nor do I say a word about sines; neither do I assume anything as to the absolute lengths of any of the lines."

"Now, may I ask you for an explicit answer to one question. How do you reconcile this with your denial (folio 8, of Letter December 9, 1868) that the chord and arc can be equal?"

"I will not pursue the subject till I receive your answer to this. But, before I close, I must refer to an extraordinary statement in your Letter of yesterday."

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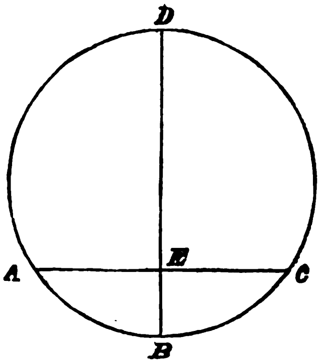
* It may be said that this reference to my Letter of the 23rd November is plain enough, and should have led me to read my copy of it. But I thought it was impossible I could have said anything capable of being construed into an argument, that I maintained the equality between a given chord and its subtending arc. Whether this reference to my Letter of the 23rd November should or should not have led me to read my copy of that communication, matters not; the fact is, I did not read it, and remained in ignorance of the lapsus on page 10, until I received Professor Whitworth's final communication.

"You say (folio 3):—' Now, the square of a line [Query—square on a line], drawn from the circumference of a circle, perpendicular to its diameter, is equal to the rectangle under the segments of the diameter. Euclid nowhere proves this, nor could he by pure Geometry, apparently: but it is very readily demonstrated by applied Mathematics.' Have you never read Euclid: Book 3: Prop. 35: of which general theorem your theorem is a particular case. And, indeed, if you read the proposition, you will find the particular case specially proved in the paragraph beginning—Secondly .... and ending with the conclusion—the rectangle B E · E D is equal to the square on A E, the very thing which you say Euclid has not proved, and (apparently) could not prove."



"Is it not rather wonderful, that the author of '*Euclid at Fault*' should be thus ignorant of the contents of the book he criticises? What will the public think of this, the next time you publish a pamphlet of correspondence?"

"Surely it is hardly necessary for you to appeal to an unnamed Member of my College, in support of your character as a Christian and a gentleman."

Believe me, my dear Sir,

Yours faithfully,

W. ALLEN WHITWORTH.

In his reply to my Letter of the 23rd November, 1868, Professor Whitworth observed:—" *However, none of your results are very startling till we get to page 21, where you assume* *without proof (and I should say you wrongly assume) that certain acute angles in your figure are* 16° 16′ *exactly. Pray how do you arrive at this conclusion? Is not the angle a very small fraction less than this? I will grant that the sines of your angles are* ·28 *and* ·96, *but I deny that the angles are exactly* 16° 16′ *and* 73° 44′." If I did not give the proof with regard to the angles here referred to, I gave him the proof, worked out by Logarithms, with reference to the angles of another right-angled triangle; and as a "*recognised Mathematician,*" he might, and ought to have convinced himself, that in a right-angled triangle of which the trigonometrical sines are ·28 and ·96, the acute angle is an angle of 16° 16′ and the obtuse angle, an angle of 73° 44′: but this did not suit his purpose. With the "Postscript," of which he attempts to make so much, I was perfectly *familiar*, and also *familiar* with the argument founded on it, which I had refuted over and over again with other opponents. This "Postscript" appeared in his second Letter in reply to mine of the 23rd of November, 1868; but at the time it did not occur to me to read the copy of my Letter of that date: indeed, there was no apparent necessity for it, since Professor Whitworth had declared that there were no results in it "*very startling*" previous to page 21. Is there nothing *startling* in an argument that would make a chord and its subtending arc equal? Was not the fact of Professor Whitworth passing over the lapsus, on page 10, as non-startling, and drawing my especial attention to the arguments on pages 21 and 23, calcaluted to divert my attention from the lapsus, upon which the "Postscript" at the end of his second Letter is founded? Was he not *conscious* that I could not have clearly expressed my meaning on page 10? Would any fair and candid controversialist have played the part that Professor Whitworth has done, with regard to that lapsus?

I am sure I need not prove to you, Sir, that the fallacy of Professor Whitworth's reasoning is based on the implied—but not expressed—assumption, that the trigonometrical functions of angles are lengths, and not ratios of one length to another. When the isosceles triangle O B C denotes one of 25 equal isosceles triangles inscribed in a circle, is not the angle BOC an angle of 14° 24′? Does it not follow of necessity, that ZOB and ZOC are angles of 7° 12′? What is the natural sine of one of these angles? Hutton makes it ·1253332. Would not Professor Whitworth, if he attempted—on his own principle of reasoning—to compute the arithmetical value of the sine of an angle of 14° 24′, make it the double of ·1253332 = ·2506664? My arithmetical value of the trigonometrical sine of this angle is ·25, and Hutton makes it less!! I cannot help thinking that you, Sir, will see at a glance, the absurdity of Professor Whitworth's reasoning.

JAMES SMITH *to* THE REV. PROFESSOR WHITWORTH.

BARKELEY HOUSE, SEAFORTH,
11*th December*, 1868.

MY DEAR SIR,

It is probably *well*, that I was very *unwell*, when I penned the concluding paragraphs of my Letter posted yesterday; for otherwise, I should certainly have replied to your Letter of the 8th inst. at greater length, and might have been tempted to give a strong expression of opinion with reference to it. Now, let us enter into a *fair* compact. For the future, let us "*stick*" to the "*mother tongue*"—in which if we are capable of reasoning, we shall be likely to reason more logically than in any other—and let us keep our temper.

Now, let 24 equal isosceles triangles be inscribed in a circle. Then : $\frac{360^\circ}{24} = 15^\circ$, and it follows, that the angles at the centre of the circle are angles of 15°. But, $\frac{15}{25} = {\cdot}6 = 36$ minutes, and *you know* that the "*recognised Mathematician*," Mr. de Morgan, has admitted that $\frac{\pi}{300}$ is the circular measure of an angle of 36 minutes, whatever be the value of π.* But, $\frac{360}{25} = 14{\cdot}4 = 14^\circ\ 24'$. Hence: $15^\circ - 36' = 14^\circ\ 24'$, which, multiplied by 24, gives $345^\circ\ 36'$, and this is a constant quantity, whatever number of equal polygons we may inscribe in a circle, and gives the ratio between the perimeter of every regular hexagon, and the circumference of its circumscribing circle. Will you venture to tell me that $\frac{3}{\pi}$ does not express the ratio between the perimeter of every regular hexagon, and the circumference of its circumscribing circle? I trow not! Well, then, by analogy or proportion, $3 : 3{\cdot}125 :: 345^\circ\ 36' : 360^\circ$, and you know as well as I do, that we *assume* the circumference of a circle to be 360°, for all practical purposes; and, since proportion *fails* with every other value of π, it follows, that $\frac{25}{8} = 3{\cdot}125$ must be the true arithmetical value of π, and the ratio of arc to chord as 15° to $14^\circ\ 24'$.

Again: $\frac{360^\circ}{6} = 60^\circ$, $\frac{60^\circ}{25} = 2^\circ\ 24'$: and $60^\circ - 2^\circ\ 24' = 57^\circ\ 36'$, which multiplied by 6 gives the constant quantity $345^\circ\ 36'$, which is the perimeter of a regular inscribed hexagon to a circle of circumference of 360°: and, by analogy or proportion, $345^\circ\ 36' : 360^\circ :: 3 : 3{\cdot}125$, and we arrive at the same result as in the previous example. Pray attempt to produce this result either with $\pi = 3{\cdot}14159$, &c., or any other false value of π!!

Well, then, whatever number of regular polygons we may inscribe within a circle, the ratio between a side of the polygon and its subtending arc is as 24 to 25, or, as $345^\circ\ 36'$ to 360°. I have given you other proofs of this in my Letter of the 7th inst., and hope I shall hear no more of your absurd charge, that I make a chord and its subtending arc equal.

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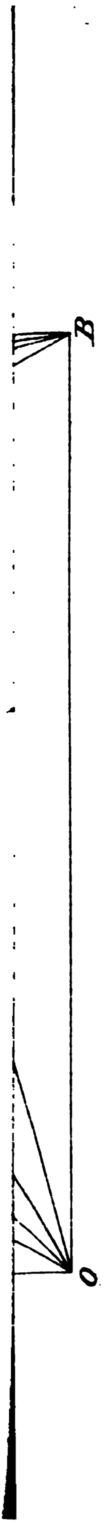
* See; *Athenæum*. Aug. 5, 1865.

In the enclosed figure (See Diagram IV.), O A E B is a quadrant, of which O A, O E, O B, &c., are radii. O C B is an equilateral triangle of which the sides are equal to the radii of the quadrant. Hence: C B is a side of a regular inscribed hexagon to a circle of which O B and O C are radii. The chords C D and D B are sides of a regular dodecagon, and the chords C E, E D, D F, and F B, are sides of a regular 24-sided polygon, to a circle of which O B and O C are radii.

It would be very absurd if I were to say that the sum of the two chords, C D and D B, or, the sum of the four chords C E, E D, D F, and F B was not greater arithmetically than the chord C B. But, C B = radius of the quadrant; therefore, $2\,\pi\,(C\,B)$ = circumference of a circle of which O B and O C are radii. Well then, by hypothesis, let the chord C E, a side of the 24-sided polygon, be represented by the *mystic* number 4. Then: the equation or identity, $24\left\{\frac{\pi}{3}(\text{chord }C\,E)\right\} = 6\left\{\frac{\pi}{3}(\text{chord }C\,B)\right\}$ and gives the circumference of the circle, whatever be the value of π. For example: By hypothesis, let $\pi = 3{\cdot}1416$. Then: $24\left\{\frac{\pi}{3}(\text{chord }C\,E)\right\} = 6\left\{\frac{\pi}{3}(\text{chord }C\,B)\right\}$; that is, $24\left(\frac{3{\cdot}1416\times 4}{3}\right) = 6\left(\frac{3{\cdot}1416\times 16}{3}\right) = 100{\cdot}5312$ = circumference of a circle, of which O A or O B is the radius, on the hypothesis that $\pi = 3{\cdot}1416$. What!—you may say—do you mean to tell me that the four chords, C E, E D, D F, and F B are, together, only equal to the chord C B? No! I don't mean to tell you anything of the kind! I have already told you it would be very absurd to do so! But will you venture to tell me that 100·5312 is *not* the circumference of a circle, of which 4 times 4 = 16 is the radius, on the hypothesis that $\pi = 3{\cdot}1416$? This is a puzzle to Mathematicians, but as "*a reasoning geometrical investigator*," you will, or ought to see, that I have given you the explanation in my Letter of the 7th inst. Well then, 100·5312 is the circumference of a circle of radius 16, on the hypothesis that $\pi = 3{\cdot}1416$. Then: $\frac{100{\cdot}5312}{24} = 4{\cdot}1888$ = the arc subtending the chord C E. Hence: $24\left\{\frac{3}{\pi}(\text{arc }C\,E)\right\} = 6\,(C\,B)$, that is, $24\left(\frac{3\times 4{\cdot}1888}{3{\cdot}1416}\right) = 6\times 16 = 96$ = the perimeter of a regular inscribed hexagon to a


B
O

circle, of which O A or O B, which are equal to C B, is the radius, when radius = 16. But any other finite and determinate hypothetical value of π will produce the same result. Well, then, all the chords in the enclosed geometrical figure are to their subtending arcs in the same ratio; and on the hypothesis that $\pi = 3{\cdot}1416$, the ratio is as 4 to 4·1888.

Now, may I ask you to describe a circle, and inscribe a regular pentagon and a regular hexagon? Euclid teaches us how to construct the pentagon, and the merest tyro in Geometry knows how to construct the hexagon. Well, then, let the circumference of the circle = 360. I only fix upon 360, because, for all practical purposes, we *assume* 360 as the circumference of the circle. We then *assume* the circumference to contain 360 units which we call degrees, and these we again sub-divide into 60 equal parts, which we call minutes. Now, it would be very absurd, if I were to say that the perimeter of the pentagon was arithmetically exactly equal to the perimeter of the hexagon. But, mark! $\frac{360}{5} = 72$: $\frac{72}{25} = 2{\cdot}88$: therefore, $72 - 2{\cdot}88 = 69{\cdot}12$. $\frac{360}{6} = 60$: $\frac{60}{25} = 2{\cdot}4$: therefore, $60 - 2{\cdot}4 = 57{\cdot}6$. Hence: $5\,(69{\cdot}12) = 6\,(57{\cdot}6)$, and this equation or identity $= 345{\cdot}6$, and is equal to the perimeter of a regular inscribed hexagon to a circle of circumference = 360: and, by analogy or proportion, $69{\cdot}12 : 72 :: 57{\cdot}6 : 60$. But, $\frac{4}{25}$ (circumference) = $\frac{3}{3{\cdot}125}$ (arc subtending a side of the hexagon); that is, $\frac{4}{25}\,(360) = \frac{3}{3{\cdot}125}\,(60)$, and this equation or indenity $= 57{\cdot}6$, and is equal to the radius of a circle of circumference 360. Hence: $\frac{3{\cdot}125}{3}\,(345{\cdot}6) = 6\left(\frac{3{\cdot}125}{3} \times 57{\cdot}6,\right)$ and this equation or indentity = 360 = circumference of the circle: and since 3 is the perimeter of a regular inscribed hexagon to a circle of diameter unity, it follows, that 3·125 is the true circumference of a circle of diameter unity. But further: $2\,\pi$ (radius) = circumference in every circle. Hence, $6{\cdot}25\,(16) = 10^2$; and this equation or identity = circumference of a circle of radius 16, = 100.*

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* Professor de Morgan disposes of this argument very summarily by such *nonsense* as the following: —"And so we are to *argue* with a man who produces a pentagon of which four sides are together *geometrically* larger than the fifth, but *mathematically* equal to it." See: *Athenæum*, September 28, 1867.

Now, $\frac{4}{25}(\pi) = \frac{4 \times 3{\cdot}125}{25} = {\cdot}5$ = radius of a circle of diameter unity. $\frac{25}{24}({\cdot}5) = 5{\cdot}2083333333333$ with 3 to infinity, and represents one sixth part of the circumference of the circle; therefore 6 (5·2083333333333) = 3·1249999999998. Surely! this must as certainly represent to us 3·125, as the never ending series 1 + $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$ + &c., represents to us the arithmetical quantity 2.

When you have given these facts your attention, I cannot conceive it possible that I shall have you again asserting that I make a certain chord and its subtending arc equal.

December 12th.

I had written so far when your Letter of yesterday came to hand.

Your reasoning is based on the assumption—implied but not expressed—that the trigonometrical functions of angles are lengths, not ratios of one length to another. Hence the fallacy of your reasoning. "*If you can't see this, I can't help it, but the fact remains notwithstanding.*"

Now, my dear Sir, I may tell you that the questions at issue between us are involved in the trigonometrical functions of angles or ratios, of which the sine and co-sine are the principle; and without the introduction of ratios into the controversy, there would be no possibility of our ever arriving at a conclusion; and I repeat, that if the hypothenuse and shortest side of a right-angled triangle be in the ratio of 8 to 1, the trigonometrical sine of the accute angle is as certainly ·125, as ·5 is the trigonometrical sine of the acute angle in a right-angled triangle of which the hypothenuse and shortest side are in the ratio of 2 to 1.

Now, $\frac{360}{25} = 14{\cdot}4 : \frac{14{\cdot}4}{25} = {\cdot}576$; therefore, 14·4 — ·576 = 13·824. Hence: 13·824 : 14·4 : : 3 : 3·125 : and 25 (13·824) = 345·6 = the perimeter of a regular inscribed hexagon to a circle of circumference = 360. $\frac{360}{24} = 15 : \frac{15}{25} = {\cdot}6$; therefore, 15 — ·6 = 14·4. Hence: 14·4 : 15 : : 3 : 3·125 : and 13·824 : 14·4 : : 14·4 : 15 and thus we get the ratio of chord to arc in both cases. And the chord subtending an angle of 15°, at the centre of a circle, is equal to the arc subtending an angle at the centre of a circle of 14° 24'.

I hardly supposed it possible that you could have mistaken my meaning, with reference to what I said Euclid had not proved. Perhaps I should have introduced the word *mathematically*. Euclid deals with Geometry, not as a branch of Mathematics, but as a pure Science. Take your own figure (No. 2) in your Letter of the 11th instant. If we put a value on the line A E, say $\sqrt{15}$, is not this an application of Mathematics to Geometry? But, having fixed the value of the line A E, how will you find the values of D E and E B by pure Geometry? Your catching at a word, and putting a query, is not worth further notice; and your application of the word *apparently* is a gross perversion of my statement.*

Will you refer to the diagram enclosed in my Letter of the 30th November *(see Diagram II.)* Join B E, and then prove that the triangle B O E thus constructed, the triangles D G H and A F M in the diagram enclosed in my Letter of the 23rd November *(see Diagram I.)*, and the triangle H P T in the diagram "*Euclid at Fault*," are *not* similar right-angled triangles; and that they have *not* the sides that contain the right angle in the ratio of 7 to 24? If you can't accomplish this—and you certainly can't—you should admit that when the $\sqrt{15}$ represents a line, it stands *not* for its extracted root = 3·872, &c., but for 3·875.

Let the circumference of a circle = 625. Then : $\frac{625}{2\pi} = \frac{625}{6{\cdot}25} =$ 100 = radius : and, 6 (radius) = 6 × 100 = 600 = the perimeter of an inscribed regular hexagon. Let the circumference of a circle = 600. Then : $\frac{600}{2\pi} = \frac{600}{6{\cdot}25} = 96$ = radius : and, 6 (radius) = 6 × 96 = 576 = the perimeter of a regular inscribed hexagon. Hence:

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* The reader will find, by turning to my Letter of the 8th December, that this refers to the following paragraph in that Letter:—"Now, according to Euclid, the square of a line drawn from the circumference of a circle perpendicular to its diameter, is equal to the rectangle under the segments of the diameter. Euclid nowhere proves this, nor could he, by pure geometry; but, *apparently*, it is very readily demonstrated by applied mathematics:" and I gave the proof.

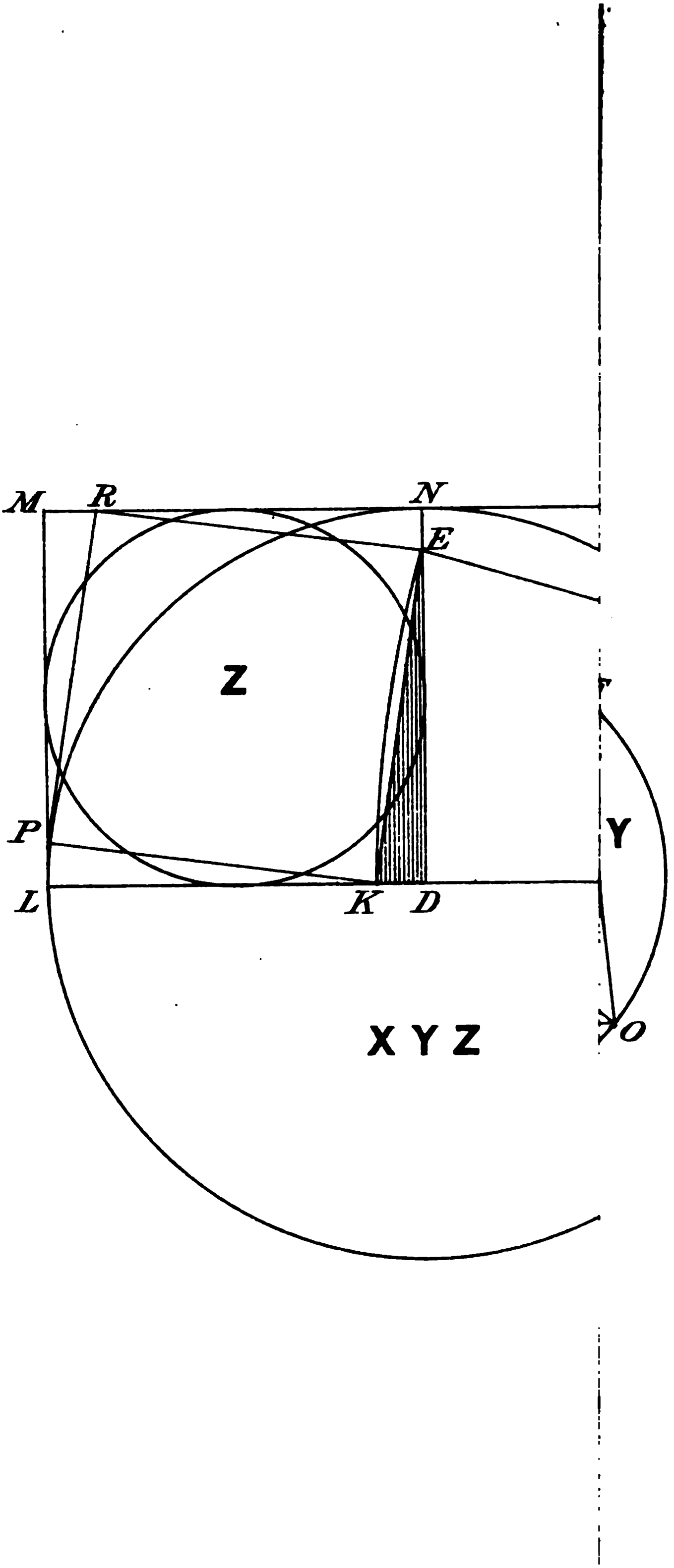
600 is a mean proportional between 625 and 576: that is, $\sqrt{(625 \times 576)} = \sqrt{360000} = 600$. Again: let the circumference of a circle = 600. Then: $\frac{600}{2\pi} = \frac{600}{6{\cdot}25} = 96$ = radius: and, 6 (radius) = 576 = the perimeter of an inscribed regular hexagon. Let the circumference of a circle = 576. Then: $\frac{576}{2\pi} = \frac{576}{6{\cdot}25} = 92{\cdot}16$ = radius: and, 6 (radius) = $6 \times 92{\cdot}16 = 552{\cdot}96$ = the perimeter of regular inscribed hexagon. Hence: 576 is a mean proportional between 600 and 552·96: that is, $\sqrt{(600 \times 552{\cdot}96)} = \sqrt{331776} = 576$. And so you might go on till you got a string of figures as *long* as from Barkeley House to Queen's College; if your life were only *long* enough to enable you to perform the *arithmetical* operation.

With reference to the concluding paragraph of your Letter I may observe, that whatever may be your opinion of my commercial morality, no intelligent and reflective-minded man could be led to any other conclusion—from a perusal of your Letters to me—than that you look upon me—mathematically—as a shuffler and trickster; and this led me to refer to a fact in connection with an unnamed member of your own college. I am prepared to reason logically, from indisputable geometrical data, to a true conclusion, on all the questions at issue between us; but you have not hitherto met me in the same spirit; and I am tempted to draw the inference that your object is not "TRUTH" but a "contention" for victory. Think, my good Sir, of the proverb of Solomon: "*He that rebuketh a man, shall afterwards find more favour, than he that flattereth with the tongue.*"

Believe me, my dear Sir,
Faithfully and kindly yours,
JAMES SMITH.

THE REV. PROFESSOR WHITWORTH.

P.S.—I shall not post this till Monday morning, and I have my own reasons for it.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JAMES SMITH *to* THE REV. PROFESSOR WHITWORTH.

BARKELEY HOUSE, SEAFORTH,
15*th December*, 1868.

MY DEAR SIR,

I might now fairly pause, and wait *your withdrawal of the absurd charge* you have brought against me, of making a certain chord and its subtending arc equal. You may probably require a little time for consideration, before you can make up your mind to this course, and in the meantime I shall pursue mine.

The geometrical figure represented by the enclosed diagram, *(see Diagram V.)* is a very remarkable and extremely interesting one. To exhaust all the properties of this figure, would require me to write more than a moderate-sized pamphlet.

CONSTRUCTION.

Construct the right-angled triangle A B C, making A B equal to ¾ (B C). With B as centre and B C as interval describe the circle X, and with C as centre and C A as interval describe the circle Y. Produce A B to meet and terminate in the circumference of the circle Y at the point H, and join H C. Produce A C and G C to meet and terminate in the circumference of the circle Y at the points O and F, and join A G, G O, O F and F A, and thus construct an inscribed square, A G O F, to the circle Y. Produce C B to D, making B D equal to 5 times B C. Produce C D to L, making D L equal to twice B C, or ⅓ (C D). On D L describe the square D L M N, and inscribe the circle Z; and with D as centre and D L or D N as interval, describe the circle X Y Z. From D N, a side of the square D L M N, cut off a part, D E, making D E equal to ⅞ (D N), and join E C, and thus construct the right-angled triangle E D C. Produce B A to meet M N a side of the square M L D N produced, at the point T, and from the point C, the centre of the circle Y, draw a straight line parallel to B T to meet M T produced, at the point V. From B T cut off a part B S, making B S equal to A C, and join S W. With C as centre and C E as interval, describe an arc E K, and join E K, and so get the chord to the arc. On the chord E K describe the square E K P R.

Now, I shall direct your attention to a number of interesting facts with reference to this geometrical figure, and then found some theorems upon it, which I shall give you for solution.

The point W is the point of intersection between the circumference of the circle X Y Z and the line L C, and it follows, that the line L C is bisected at the point W. The triangle A B C, the generating figure of the diagram, represents the *primary* commensurable right-angled triangle; that is to say, it is derived from the consecutive numbers 1 and 2; A B denoting the sum, and B C twice the product of these numbers: and it follows, that when A B = 3, B C = 4, and A C = 5. Again: the triangle S B W is a commensurable right-angled triangle, and is derived from the consecutive numbers 2 and 3: and it follows, that when S B = 5, B W = 12, and S W = 13. Again: the triangle E D C is a commensurable right-angled triangle, and is derived from the consecutive numbers 3 and 4: and it follows, that when E D = 7, D C = 24, and E C = 25: and it also follows, that E C, the hypothenuse, is equal to 5 times A C, the hypothenuse of the right-angled triangle A B C, and E D the perpendicular, equal to the sum of D C and E C, when E D = 7. Again: the circles Z and X, and the squares R P K E and A G O F, are exactly equal in superficial area: and it follows, that because the circle X and the square A G O F are equal, the area of a circle of radius 4 is equal to the area of an inscribed square to a circle of radius 5. Again: The area of the rectangle T B C V is exactly equal to the area of inscribed squares to the circles X and Z: and it follows, that the sum of the areas of the rectangles N D B T and T B C V is exactly equal to the area of a regular inscribed dodecagon to the circle X Y Z: and the line D C equal to the perimeter of a regular inscribed hexagon to the circles X and Z.

I might direct your attention to many other facts; but, it appears to me, no further information should be necessary to enable you to solve the following theorems:—

THEOREM 1.

Let the area of the circle Z or X be represented by any arithmetical quantity, say 10368, or any other quantity if you like it better. I have a reason for selecting this quantity, and should be glad if you would adopt it. Find the arithmetical value of the line E C, the hypothenuse of the right-angled triangle E D C, and prove that it is equal to the circumference of the circle X or Z.

THEOREM 2.

Let E C the hypothenuse of the right-angled triangle E D C be represented by any arithmetical quantity, say $\sqrt{1000}$, or any other arithmetical quantity if you like it better. In this case I have no choice whatever. Find the circumference of the circle X Y Z, and prove that it is equal to 10 times A C the hypothenuse of the right-angled triangle A B C.

THEOREM 3.

Let the area of the square A G O F be represented by any arithmetical quantity, say 60.* Find the area of the circumscribing circle Y, and the arithmetical values of the sides of the right-angled triangle A B C, the hypothenuse of which is the radius of the circle Y, and prove that the sum of the squares of the three sides of the triangle A B C, is equal to the area of the circle X.

Now, my dear Sir, will you be kind enough to favour me with the solution of these three theorems? You have made an onslaught on my *geometrical capacity*, and I have met you by fair and legitimate argument. I think you cannot refuse me this favour, with any shew of reason; and by furnishing the solutions, you will at once establish your *geometrical and mathematical capacity.* Grant me this favour, and I assure you I will give you no further trouble in the way of solving theorems, although I may bring under your notice many other beauties of geometry.

In conclusion, I may assure you that the solution of these theorems is very simple, when we have got hold of the right way to go about the solution of them.

Believe me, my dear Sir,
Faithfully yours,
JAMES SMITH.

THE REV. PROFESSOR WHITWORTH.

I may tell *you*, Sir, as a fact, that I have never been able to induce any of my correspondents to construct a problem, or solve a theorem; and it will be as manifest to you, as it is to me, that Professor Whitworth never attempted to solve the problem I suggested to him, in my Letter of the 4th December. I may tell you another fact. When I have constructed a problem, and by means of it solved a theorem, proving $\pi = \frac{25}{8} = 3{\cdot}125$, my correspondents have invariably found it convenient to evade my arguments, and to avoid grappling with my demonstrations. The Mr. R—— referred to in the early part of my pamphlet, "*Euclid at fault*," went the length of telling me, that we can prove nothing by practical or constructive Geometry. If so, what value or utility is there, or can there be, in the science of Geometry?

The diagram No. III. is a fac-simile of that enclosed in my Letter of the 7th December to Professor Whitworth, with the addition of the three straight lines y P, y F, y V, and the circle Y Z, and the following may be adopted as the method of constructing the diagram.

Let A and B denote two points dotted at random. Join these points, and on A B describe the equilateral triangle O A B. From the angle O draw a straight line, bisecting the angle and its subtending chord at the point H: and from the point H draw a straight line, parallel to the side O A in the triangle O A B, to meet and bisect the side O B at the point K. With O as centre and O K as interval, describe the circle P: with O as centre and O H as interval, describe the circle N: and with O as centre and O B as interval, describe the circle X. From the angle B in the equilaterial triangle O A B, draw a straight line at right angles to O B, and therefore tangental to the circle X, to a point C, making B C equal to ¾(O B), and join O C. With O as centre and O C as interval, describe the circle Y. Produce C B to meet and terminate in the circumference of the circle Y at the point P, and join O P. It is obvious that O B C and O B P are similar and equal right-angled triangles, and have the sides that contain the right angle in the ratio of 4 to 3, by construction. From the angle C in the triangle O B C, draw a straight line C D, at right angles to O C, making C D equal to O C, join O D, and with C as centre and C O or C D as interval, describe the arc O D. Produce D C to meet O B produced at the point F, and thus construct the right-angled triangle O C F, and join P F. With O as centre and O F as interval, describe the circle M: and with O as centre and O D as interval, describe the circle Z. Produce O C to meet and terminate in the circumference of the circle Z at the point E: and from the point O the centre of the circles, draw a straight line at right angles to O E, to meet and terminate in the circumference of the circle Z at the point N: and join D E and D N. Produce E O to meet and terminate in the circumference of the circle Z, at the point L: and from the point D draw a straight line parallel to E L to meet and terminate in the circumference of the circle Z at the point M, and join M L and M N. It is self-evident, that L M, M N, N D, and D E, are sides of an inscribed regular octagon to the circle Z. From the angle F, in the right-angled triangle O C F, draw a straight line at right angles to O F, and therefore tangental to the circle M, to meet O E produced at the point R, and thus construct the right-angled triangle O F R. With O as centre and O R as interval, describe the circle X Z. From the angle R in the triangle O F R, draw a straight line at right angles to O R, and therefore tangental to the circle X Z, to meet O F produced, at the point V. Produce R F to meet and terminate in the circumference of the circle X Z, at the point *y;* or, we might say, produce R F to meet O P produced, at the point *y*, and join *y* V, and thus construct the quadrilateral O *y* V R. Bisect O V at *x*, and with F as centre, and F *x* as interval, describe the circle Y Z. Produce N O to meet and terminate in the circumference of the circle Z, at the point T. From the point *e*, draw a straight line at right angles to O *e*, and therefore tangental to the circle X, to meet and terminate in the line O N at the point *f*, and from the point O, draw a straight line at right angles to O *e*, to meet and terminate in the circumference of the circle X at the point *g*, and join *g f*, and thus construct the square O *e f g*. It is self-evident, that L E and N T, diameters of the circle Z, at right angles to each other, intersect the circumference of the circle X, at the points *a*, *b*, *c*, and *d*. Join *a b*, *b c*, *c d*, and *d a*, and thus construct an inscribed square to the circle X. In the square *a b c d* inscribe the circle X Y : and in the circle X Y inscribe the square *a′ b′ c′ d′*. It is self-evident that *a b c d* is an inscribed square to the circle X. It is also self-evident, that O *f*, O *d*, and O *d′*, are the diagonals of squares. It is also self-evident that L E and N T, which are diameters of the circle Z, at right angles to each other, intersect the circumference of the circle M, at the points *n*, *o*, *p*, and *m*. Join *n o*, *o p*, *p m*, and *m n*, and thus construct the square *m n o p*. It is obvious or self-evident, that *m n o p* is an inscribed square to the circle M.

Now, O P F C is a quadrilateral, and the sides O P and O C are to the sides P F and C F in the ratio of 4 to 3, by construction: and O P and O C are radii of the circle Y. Hence: it may be demonstrated that the square *m n o p*, and the circle Y, which is partly within and partly without the square *m n o p*, are exactly equal in superficial area; and that both are equal to the sum of the squares of the four sides of the quadrilateral O P F C; and also equal to the sum of the squares of the diagonals of the quadrilateral O P F C, together with four times the square of the line that joins the middle points of the diagonals.

When O K, the radius of the circle P, $= 2$, then, O B the radius of the circle $X = 4$; and O C the radius of the circle $Y = 5$. But, $OP = OC$, for they are radii of the circle Y; therefore, $\frac{3}{4}(OP)$ or $\frac{3}{4}(OC) = \frac{3 \times 5}{4} = \frac{15}{4} = 3{\cdot}75 = PF$ and CF; therefore, $(OP^2 + OC^2 + PF^2 + CF^2) = 3\frac{1}{8}(OP^2)$ or $3\frac{1}{8}(OC^2)$; that is, $(5^2 + 5^2 + 3{\cdot}75^2 + 3{\cdot}75^2) = 3\frac{1}{8}(5^2)$; or, $(25 + 25 + 14{\cdot}0625 + 14{\cdot}0625) = 3{\cdot}125 \times 25 = 78{\cdot}125 =$ area of the circle Y; and is exactly equal to the area of the square *m n o p*. Proof: The square *m n o p*, is an inscribed square to the circle M; and, O F the radius of the circle $M = 6{\cdot}25$, when O K the radius of the circle $P = 2$. By hypothesis, let the area of the square *mnop* $= 78{\cdot}125$. Then: $\left\{\left(78{\cdot}125 + \frac{78{\cdot}125}{4} + \frac{1}{4}\left(78{\cdot}125 + \frac{78{\cdot}125}{4}\right)\right\} = \{(78{\cdot}125 + 19{\cdot}53125) + \frac{1}{4}(97{\cdot}65625)\} = (97{\cdot}65625 + 24{\cdot}4140625) = 122{\cdot}0703125 =$ area of the circle M, and is equal to $3\frac{1}{8}(OF^2)$: and, $\frac{122{\cdot}0703125}{{\cdot}78125}$, or, $(122{\cdot}0703125 \times 1{\cdot}28) = 156{\cdot}25 =$ area of a circumscribing square to the circle M. But, $\frac{156{\cdot}25}{2} = 78{\cdot}125$, and it follows of necessity, that the square *m n o p*, and the circle Y, are exactly equal in superficial area.

Let O K the radius of the circle P = 2.

Then:

O R the radius of the circle X Z = 7·8125.

$\therefore OR^2 = 7{\cdot}8125^2 = 61{\cdot}03515625.$

$FR = \frac{3}{5}(OR) = \frac{3}{4}(OF) = 4{\cdot}6875.$

$\therefore 2(FR) = Ry = 9{\cdot}375.$

$\therefore Ry^2 = 9{\cdot}375^2 = 87{\cdot}890625.$

$RV = \frac{3}{4}(OR) = \frac{3}{5}(OV) = 5{\cdot}859375.$

$\therefore RV^2 = 5{\cdot}859375^2 = 34{\cdot}332275390625.$

$OV = 9{\cdot}765625.$

$\therefore OV^2 = 9{\cdot}765625^2 = 95{\cdot}367431640625.$

$\frac{1}{2}(OV) = Ox.$

$\therefore Ox = \frac{9{\cdot}765625}{2} = 4{\cdot}8828125.$

$FV = \frac{3}{4}(RF) = \frac{3}{5}(RV) = 3{\cdot}515625.$

$\therefore xF = OF - Ox = 6{\cdot}25 - 4{\cdot}8828125 = 1{\cdot}3671875.$

$\therefore 4(xF^2) = 4(1{\cdot}3671875^2) = 4(1{\cdot}869201660156 25) = 7{\cdot}4768066406 25.$

But, Oy V R is a quadrilateral, and "*in any quadrilateral the sum of the squares of the four sides is equal to the sum of the squares of the diagonals, together with four times the square of the line joining the middle points of the diagonals.*" Now, x F is the line that joins the middle points of the diagonals, in the quadrilateral Oy V R.

Hence:

$\{OV^2 + Ry^2 + 4(xF^2)\}$ = the sum of the squares of the four sides of the quadrilateral OyV R; that is, $(95{\cdot}367431640625 + 87{\cdot}890625 + 7{\cdot}4768066406 25) = (61{\cdot}03515625 + 34{\cdot}332275390625 + 34{\cdot}332275390625 + 61{\cdot}03515625)$, and this equation $= 190{\cdot}73486328125 = 3\frac{1}{8}(OR^2)$ = area of the circle X Z. But, $2(190{\cdot}73486328125) = 4(OV^2)$; and it follows, that twice the sum of the squares of the four sides of the quadrilateral OyV R, is equal to the area of a circumscribing square to a circle of which O V is the radius. *No other* *value of* π, *but* $\frac{25}{8}$ *can be made to harmonize with these geometrical truths; and it follows, that* 3·125 *is the true arithmetical value of* π, *and makes* 8 *circumferences* = 25 *diameters in every circle.*

THEOREM.

Let the area of the circle P be represented by any arithmetical quantity, say 60. Find the area of a regular inscribed dodecagon to a circle of which O V is the radius, and prove that it is exactly equal to 6 (O V × O *x*.)

It will be as obvious to you, Sir, as it is to me, that whether the arithmetical value of π be *indeterminate*, in the *common sense* acceptation of the meaning of the word *indeterminate;* or, *finite* and *determinate*, in the sense attached to these words by the Reverend Professor Whitworth, that in either case the solution of this theorem would be an impossibility. But, I maintain that the solution of this theorem, is not impossible, and this I shall prove: and, it will become your duty as the President of "*The British Association for the Advancement of Science*," and a professional Mathematician, to controvert my proof if you think it untenable ; and if not, it is equally plain, that as an *honest man*, your duty is to admit the truth of my solution.

Referring to Diagram III., I may observe:—It is self-evident that with O as centre, and O V as interval, we may describe another circle, and this circle may be denoted by the symbol β. Conceive this addition to be made to the diagram. Now, when O K the radius of the circle P = 2, then, O V the radius of the circle β = 9·765625. But, $3\frac{1}{8}\,(9{\cdot}765625^2) = (3\frac{1}{8})^5$: and this equation or identity = 298·023223876953125 = area of the circle

β, when O K the radius of the circle P = 2. Now, $\frac{\pi \text{O V}^2}{\pi \text{O K}^2}$ gives the number of times the area of the circle P is contained in the area of the circle β: but, π goes out, and it follows, that $\frac{\text{O V}^2}{\text{O K}^2} = \frac{9{\cdot}765625^2}{2^2} = \frac{95{\cdot}367431640625}{4} =$ 23·84185791015625, is the number of times the area of the circle P is contained in the area of the circle β. Hence: $12\frac{1}{2}$ (23·84185791015625) = $3\frac{1}{8}$ (O V^2); that is (12·5 + 23·84185791015625) = (3·125 × 95·367431640625): and this equation or identity = 298·0232238769553125 = area of the circle β, when O K the radius of the circle P = 2.

Well, then, by hypothesis, the area of the circle P = 60, and it follows of necessity, that 60 (23·8418579101·5625) = 1430·511474609375 = area of the circle β. But, $\frac{24}{25}$ (1430·511474609375) = $\frac{24 \times 1430{\cdot}511474609375}{25} =$ 1373·291015625 : or, by analogy or proportion, 3·125 : 3 :: 1430·511474609375 : 1373·291015625 : and it follows, that 1373·291015625 is the area of a regular inscribed dodecagon to the circle β.

Proof: $\sqrt{\frac{\text{area}}{\pi}}$ = radius in every circle, whatever be the value of π: and it is self-evident, that π can but have one true arithmetical value. Now, $\sqrt{\left(\frac{1430{\cdot}511474609375}{3{\cdot}125}\right)} = \sqrt{(457{\cdot}763671875)}$ = O V the radius of the circle β on the THEORY that 8 circumferences = 25 diameters in every circle. $\frac{\text{O V}}{2} = \frac{\sqrt{457{\cdot}763671865}}{2} = \sqrt{(114{\cdot}44091796875}$ = semi-radius of the circle β: and, 6 (radius × semi-radius) = area of an inscribed regular dodecagon to every circle; and it follows, that 6 ($\sqrt{457{\cdot}763671875} \times \sqrt{114{\cdot}44091796875}$)

$=6(\sqrt{52386\cdot89482212066650390625})=(6\times228\cdot8818359375)$ $= 1373\cdot291015625$ = area of a regular inscribed dodecagon to the circle β; that is, to a circle of which O V is the radius.

Now, Sir, neither you nor any other living Mathematician, can produce this result with any other value of π, but that which makes 8 circumferences = 25 diameters in every circle. This makes $\frac{25}{8} = 3\cdot125$ the true arithemetical value of π. It would appear that these truths are beyond the reach of the capacity of the existing race of *professional* Mathematicians. It may not be in my day, but I venture to tell you, Sir, that the day will come, when these truths will be recognised, by Mathematicians; and will be known to, and comprehended by, every first-class school-boy !!

THE REV. PROFESSOR WHITWORTH *to* JAMES SMITH.

LIVERPOOL, *December* 15*th*, 1868.

MY DEAR SIR,

Your words (December 8th) are:—"The square of a line drawn from the circumference of a circle perpendicular to its diameter, is equal to the rectangle under the segments of the diameter. Euclid no where proves this, nor could he, by pure Geometry, apparently."*

I have shewn you that he does prove it explicitly and absolutely, and you now say that he does not prove it "mathematically." You

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* By referring to my Letter of the 8th, the reader will observe, that *my words were*:—Euclid no where proves this, nor could he by pure Geometry: but, *apparently*, it is readily demonstrated by the aid of Mathematics.

must give a strange definition to the word Mathematics. Please say what your definition is, that I may understand you for the future? You appear to mean that Euclid does not express the perpendicular and the segments arithmetically. It is well that he does not, for if he did, his proofs, like yours, would only apply to a particular case. But he proves, *absolutely*, that the square and rectangle, defined in your enunciation, are equal, geometrically and absolutely; and therefore they must be equal, whatever be the common unit of area in which any one may be pleased to express them.

Can you say that, when you penned the paragraph I have quoted, you were conscious of Euclid's proof to which I have referred you?

The proofs you now send me are just as invalid as those I have already criticised. But I decline to discuss them until we have settled the previous ones. I ask you to answer the question in my previous Letter.

Does it not follow, from the proof you sent in a former Letter, that the circumference of a xxv.-gon is equal to the circumference of its circumscribing circle? If not, where is the false step in my inference? If this equality stand, do you not maintain the equality of an arc with its chord? *

You seem to think that there is something personal in a proof being satisfactory. And that if I object to a proof, you may pass it by and put forth another. Please observe that I never object to a proof without shewing where its argument fails, and if it fails to me, it must fail to all others who have a logical mind, unless, indeed, I am falling into some misapprehension, which you can correct. There can be no question of opinion whether any step in an argument is logical or not, at least in a mathematical argument. When you put forth a proof you must answer any objection to it, or admit it to be unsound. Instead of this you try to pass it over, and replace it by another and another.

You speak of the perimeter of a hexagon being so many *degrees*. You may use "degree" or any other word in any sense you please, provided you adhere to that sense. I always mean by a degree the

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* This refers to the paragraph in my Letter of November 23rd, 1868, in which I have admitted that there is a lapsus; but the reader will observe that in this paragraph Professor Whitworth makes no reference to the date of that Letter.

90th part of a right angle.[*] You sometimes mean the length of a line. Could you not use a different symbol to distinguish what you would call a linear degree?

I have no objection to you speaking of a circumference as divided into 360, or 380, or any other number of equal parts.

I observe, that your work with recurring decimals is wrong, as usual, but I reserve this till I come in due course to these proofs. At present we must settle the question of the equality of chord and arc.[†]

Faithfully yours,
W. ALLEN WHITWORTH.

JAMES SMITH *to* THE REV. PROFESSOR WHITWORTH.

BARKELEY HOUSE, SEAFORTH,
17th December, 1868.

MY DEAR SIR,

I *admit* that you are a "*recognised Mathematician;*" and I *assume* that you are "*a reasoning geometrical investigator,*" and an *honest* man.

Under these circumstances, should you not point out the flaw—if there be a flaw—in the construction of the geometrical figure represented by the diagram enclosed in my Letter of the 15th inst.? Should you not point out the error and prove it—if there be an error—in what I state to be facts, with reference to the said geometrical figure?

[*] I also mean, by a degree, the 90th part of a right angle. Is not a degree the 360th part of 4 right angles? Is not the circumference of a circle a line? What matters it, then, whether a line be straight or curved, so far as regards dividing it into equal parts, and expressing the value of these parts in degrees?

[†] Is there not a distinction between a recurring and a repeating decimal? According to the Imperial Dictionary—to which one of my Correspondents referred me as an authority on such matters: $\frac{1}{7} = \cdot142857142857$, is a recurring decimal, and $\frac{1}{3} = \cdot3333$ with 3 to infinity is a repeating decimal.

If you are "*a reasoning geometrical investigator*" and an *honest* man, should you not, as a "*recognised Mathematician*," solve the theorems founded upon the said geometrical figure?

If you are an *honest* man, and find yourself incompetent to solve the theorems I gave you for solution, in my Letter of the 15th inst., should you not admit the fact?

The common sense of mankind has decided, that no man shall constitute himself judge and jury in his own cause; and no such *absurdity* can be admitted, as one of the laws of fair controversy!

Now, my good Sir, bear in mind that it was not I who attacked you, but you who attacked me; branding me as a *maligner* and *libeller* of "*recognised Mathematicians.*" It is for *me* to defend myself from these *foul* charges; and the "*onus probandi*" rests with *you* to prove that my defence is untenable!!

Faithfully yours,

JAMES SMITH.

THE REV. PROFESSOR WHITWORTH.

P.S.—5-45 p.m. Yours of this morning to hand.

THE REV. PROFESSOR WHITWORTH *to* JAMES SMITH.

16, PERCY STREET, LIVERPOOL,
December 17*th*, 1868.

MY DEAR SIR,

I have made no absurd charge. But you tell me that the perimeter of a xxv.-gon is $\frac{25}{8}$ of the diameter of the circumscribing circle, and the circumference of the circle is also $\frac{25}{8}$ of the diameter. You must either admit one of these statements to be wrong, or else maintain the equality of a chord and arc. I ask you which of the statements is incorrect, and you persist in trying to change the subject. I shall take no notice of any more of your letters until you tell me explicitly how much of your ground you abandon,—whether you still maintain the side of the xxv.-gon to be $\frac{1}{8}$ of the diameter, and its perimeter, therefore, $\frac{25}{8}$ of the diameter—and whether you still maintain the circumference to be $\frac{25}{8}$ of the diameter.

I also ask you again, whether you were conscious of Euclid's proof, Book 3: Prop. 35: Case 2: when you said that he had not proved, and could not prove, that case.

If you were conscious of it, and were depending on the quibble with which you try to defend yourself, I am sorry for you. If you were ignorant of it, I would respectfully suggest that you should read Euclid before you publish any more pamphlets on "*Euclid at Fault.*"

You invite me, in your quotation from Solomon, to speak plainly in reproof of the course you have taken. But I prefer to reserve reproof for knaves and fools, and to meet ignorance with instruction.

Believe me,
Very truly yours,
W. ALLEN WHITWORTH.

By no exercise of ingenuity on my part could I have demonstrated more effectually, than Professor Whitworth has himself proved in his Letter of the 11th December, that, in the postscript to his second Letter of 28th November, to which he has referred in every subsequent communication, he treats the trigonometrical functions of angles as lengths, not as ratios of one length to another: and when he asserts in his Letter of the 15th December, that he has "*made no absurd charge,*" what could he refer to, if not to the lapsus on page 10 of my Letter of the 23rd November? It appears to me he forgets, or ignores the fact, that the sine of an arc is half the chord of twice the arc, and that it necessarily follows, that the ratio between the sine of an angle and the sine of its arc is a varying ratio, at every doubling of the sides of a polygon inscribed in a circle.

JAMES SMITH *to* THE REV. PROFESSOR WHITWORTH.

BARKELEY HOUSE, SEAFORTH,
19th, December, 1868.

SIR,

If your mind is impervious to reason, and you are imcompetent to grapple with argument, or to solve a geometrical theorem, the sooner you cease to trouble me with your Letters the better; not only for my convenience, but probably, for your own mathematical reputation. You *know* that I had no wish to enter into a correspondence with you; but, YOU, my good Sir, having forced me into one, I shall now make use of you for my own purposes: that is to say, I shall continue to address Letters to you with a view to publication, and when I do publish, you may rest assured, I shall give your Letters in their "entirety;" and the *common sense* of mankind will, in the long run, decide which of us has *contended* for "*TRUTH*," and which for "*victory*."

I shall now proceed to solve the first of the theorems given in my Letter of the 15th inst.

If 24 equal isosceles triangles be inscribed in a circle, the angles at the centre of the circle are angles of 15°, and are subtended by arcs of 15°, and these arcs are equal to 15, when the circumference of the circle = 360. If 25 equal isosceles triangles be inscribed in a circle of circumference = 360, the angles at the centre of the circle are angles of 14° 24′, and are subtended by arcs of 14° 24′, and these arcs, decimally expressed, are equal to 14·4. The angle at the centre of a circle, subtended by an arc equal to radius, is an angle of 57° 36′, and this arc = 57·6, when expressed decimally. Hence: $24(57{\cdot}6 \times \frac{15}{2}) = 25(57{\cdot}6 \times \frac{14{\cdot}4}{2})$; that is, $24(57.6 \times 7{\cdot}5) = 25(57{\cdot}6 \times 7{\cdot}2)$, and this equation or indenty = 10368.

Now, let the area of the circle Z = 10368, and be given to find the arithmetical value of the line E C, the hypothenuse of the right-angled triangle E D C, in the diagram inclosed in my Letter of the 15th inst., and prove that it is equal to the circumference of the circle X or Z.

Then: $\sqrt{\frac{10368}{3·125}} = \sqrt{3317·76} = 57·6$ = radius of the circle Z. 6 (radius) = $(6 \times 57·6) = 345·6$ = the perimeter of a regular inscribed hexagon to the circle Z, and is equal to D C the base of the right-angled triangle E D C. $\frac{7}{24}$ (D C) $= \frac{7 \times 345·6}{24} = \frac{2419·2}{24} = 100·8$ = E D, the perpendicular of the right-angled triangle E D C; therefore, $(\text{E D}^2 + \text{D C}^2) = (100·8^2 + 345·6^2) = 10160·64 + 119439·36) = 129600$ = E C^2; therefore, $\sqrt{129600} = 360$ = E C, the hypothenuse of the right-angled triangle E D C. But, E C = 5 times A C, the hypothenuse of the right-angled triangle A B C, by construction; therefore, $\frac{\text{E C}}{5} = \frac{360}{5} = 72$ = A C : B C = $\frac{4}{5}$ (A C), by construction; therefore, $\frac{4}{5}$ (A C) $= \frac{4 \times 72}{5} = \frac{288}{5} = 57·6$ = B C : A B = $\frac{3}{5}$ (A C) by construction; therefore, $\frac{3}{5}$ (A C) $= \frac{3 \times 72}{5} = \frac{216}{5} = 43·2$ = A B. Now, 2π (radius) = circumference in every circle; therefore, 2π (B C) = $6·25 \times 57·6 = 360$ = circumference of the circle X, and is exactly equal to the hypothenuse of the right-angled triangle E D C, (Q.E.D). But further, $(\text{A B}^2 + \text{B C}^2 + \text{A C}^2) = 3\frac{1}{8} (\text{B C}^2)$, that is, $(43·2^2 + 57·6^2 + 72^2) = 3·125 (57·6^2)$, or, $(1866·24 + 3317·76 + 5184) = (3·125 \times 3317·76) = 10368$ = area of the circle X, and is exactly equal to the given area of the circle Z; and it follows, that the circumferences of the circles X and Z are equal, and both equal to the hypothenuse of the right-angled triangle E D C.

Let the area of the circle Z, $= \frac{3·125}{4} = ·78125$. Then: Any first-class school-boy will be able to work out the calculations, and prove that E C the hypothenuse of the right-angled triangle E D C, is exactly equal to the circumference of a circle of diameter unity: and it is therefore unnecessary to burden my Letter with the calculations.

I have pointed out in my Letter of the 15th inst., that when the sides of the triangle E D C are 7, 24 and 25, then, E D^2 = the sum of D C and E C, that is, $7^2 = (24 + 25) = 49$. This is a particular case and quite unique; but, it does not affect the *theory* of commensurable right-angled triangles, derived from two consecutive numbers. The sides containing the right angle in the triangle

E D C, are in the ratio of 7 to 24, and the triangle is a commensurable right-angled triangle, whatever arithmetical value we may assume as the hypothenuse, or either of the sides containing the right angle.

Again: $\left(\frac{360}{10} \times \frac{3{\cdot}125}{4}\right) = \frac{1125}{40} = 28{\cdot}125$. Now, when the radius of the circle X = 4, and the radius of the circle Y = 5, the circle X and the square A G O F are exactly equal in superficial area: and 28·125 represents the difference between the area of the square A G O F, and the area of the circumscribing circle Y. This, again, is a "*particular case*," and quite unique; but is in harmony with the following general law on the geometry of the circle. If A denote the area of a square, then, $\left\{\left(A + \frac{A}{4}\right) + \frac{1}{4}\left(A + \frac{A}{4}\right)\right\} =$ area of a circumscribing circle. Now, let B C the radius of the circle X, be represented by the *mystic* number 4. Then $3\frac{1}{8}(4^2) = 3{\cdot}125 \times 16 = 50 =$ area of the circle X. But, the circle X, and the square A G O F, are equal in superficial area; therefore, $\left\{\left(50 + \frac{50}{4}\right) + \frac{1}{4}\left(50 + \frac{50}{4}\right)\right\} = (62{\cdot}5 + 15{\cdot}625) = 78{\cdot}125 = 3\frac{1}{8}(AC^2)$: and in this particular and unique case, the difference between the area of the circle Y, and the area of the inscribed square A G O F $= (78{\cdot}125 - 50) = \frac{360}{10}\left(\frac{\pi}{4}\right) = 28{\cdot}125$. Now, this is true only, when the radius of the circle X = 4, and the radius of the circle Y = 5. But, if A denote the area of the square A G O F, and be represented by any finite arithmetical quantity, then, $\left\{\left(A + \frac{A}{4}\right) + \frac{1}{4}\left(A + \frac{A}{4}\right)\right\} = 3\frac{1}{8}(AC^2)$, and this equation or indentity = area of the circle Y circumscribed about the square A G O F, whatever finite arithmetical value we may put upon A. But further: This *particular and unique case*, is in harmony with the following general law on the geometry of the circle. If A denote the area of a circle, then $\left\{\left(A - \frac{A}{5}\right) - \frac{1}{5}\left(A - \frac{A}{5}\right)\right\} =$ area of an inscribed square. For example:—Let A denote the area of the circle Y, and be represented by the arithmetical quantity 100. Then: $\left\{\left(A - \frac{A}{5}\right) - \frac{1}{5}\left(A -\right.\right.$

$\left.\left.\frac{A}{5}\right)\right\} = \left\{\left(100 - \frac{100}{5}\right) - \frac{1}{5}\left(100 - \frac{100}{5}\right)\right\} = (80 - 16) = 64 =$ area of the inscribed square A G O F. 2 (64) = 128 = area of a circumscribing square to the circle Y. But, the area of a circle is found by multiplying the area of a circumscribing square by $\frac{\pi}{4}$; therefore, $128 \times \frac{3{\cdot}125}{4} = 128 \times {\cdot}78125 = 100 =$ the given area of the circle. Or, since $\frac{1}{{\cdot}78125}$ and $\frac{1}{1{\cdot}28}$ are equivalent ratios, it follows, that $\frac{128}{1{\cdot}28} = 100 =$ the given area of the circle.

Can you, Sir, pretend to be "*a reasoning geometrical investigator*," and fail to perceive that these facts establish, beyond the possibility of dispute or cavil by any *honest* Mathematician, that $\frac{25}{8} = 3{\cdot}125$ is the true arithmetical value of π, and makes 8 circumferences of a circle exactly equal to 25 diameters?

Now, we can conceive 360 equal isosceles triangles to be inscribed in a circle. In this case, if the circumference of the circle be represented by 360, the angles at the centre of the circle are angles of 1°, and are subtended by arcs of 1°, and, dispensing with the symbol that represents a degree, the arithmetical value of the arc is 1; and I presume you know as well as I do, that radius, multiplied by half the arc contained by two radii, is equal to the area of that part of the circle contained by the two radii. Hence: $360\,(57{\cdot}6 \times {\cdot}5) = (360 \times 28{\cdot}8) = 10368$. But, $360\,(57{\cdot}6 \times {\cdot}5) = 3\frac{1}{8}\,(57{\cdot}6^2)$; that is, $(360 \times 28{\cdot}8) = (3{\cdot}125 \times 3317{\cdot}76)$, and this equation or indentity = 10368 = area of a circle of which the circumference is 360. But, 28·8 is the semi-radius of the circle. Does not circumference × semi-radius = area in every circle?

Do you not know that a line, of any length, will enclose a larger area in the form of a circle than in any other form whatever? Where will you be, if you try to find the area of a circle of which the circumference is 360 units in length, with the mysterious $\pi = 3{\cdot}14159$, &c.? I may tell you that you can't find the area of the circle; and when you have found what you *fancy* to be an approximation to it, you will find you make it much less than 10368.

Again: $\frac{1}{25} = {\cdot}04$; and, $1 - {\cdot}04 = {\cdot}96$, which, multiplied by 360, gives 345·6, and is equal to the perimeter of an inscribed regular hexagon, to a circle of which the circumference is 360 units in length. Hence: by analogy or proportion, ·96 : 1 : : 345·6 : 360; or, ·96 : 1 : : 3 : 3·125; or, 345·6 : 360 : : 3 : 3·125; and it follows, that the sides of a 360-sided regular polygon, inscribed in a circle, are to their subtending arcs in the ratio of ·96 : 1, or in the equivalent ratios of 345·6 : 360, or, 3 : 3·125. Now, Sir, it is not I who do anything so absurd as make a certain chord and arc equal, and the perimeter of a certain polygon and circumferenee of its circumscribing circle equal: it is you who commit a gross absurdity. Your reasoning is based on the assumption that the trigonometrical functions of angles are lengths, and not ratios of one length to another. I dare say you cannot see this; but that I can't help; nevertheless, this fallacious assumption lies at the foundation of all your reasoning. Hence, the absurdity of your arguments and conclusions on the Geometry of the Circle.

Referring you to the diagram enclosed in my Letter of the 15th instant *(see Diagram V.)*, I may observe:—The acute angles in the four similar right-angled triangles about the square R P K E, are equal to half the acute angle in the right-angled triangle E D C. The angle H C G, at the centre of the circle Y, contained by the radii C H and C G, is exactly equal to the acute angle in the triangle E D C. Hence: the angles A C B, B C H, and H C G, are together equal to the right angle A C G. Now, Sir, if you are competent to grapple with the subject, you may readily convince yourself that the angles A C B and B C H are angles of 36° 52′, and the angle H C G an angle of 16° 16′; but it will not be by assuming the infallibility of our Mathematical Tables of natural sines, co-sines, &c., that you can accomplish this! You will have to find the log-sines and log-co-sines of these angles, without the aid of tables.

Your latter communications forcibly remind me of another proverb of Solomon's:—"*Seest thou a man wise in his own conceit? There is more hope of a fool than of him.*"

Your last Letter commences thus:—"*I have made no absurd charge. But you tell me that the perimeter of xxv.-gon is* $\frac{25}{8}$ *of the diameter of the circumscribing circle, and the circumference of the circle is also* $\frac{25}{8}$ *of the diameter.*" (I never told you anything of the sort. It is a gross perversion of what I have told you, and I find it difficult to persuade myself that you don't know it.) You then say:—"*You must either admit one of these statements to be wrong, or else maintain the equality of a chord and arc.*" (I of course admit one of these statements to be wrong; but the statement that is wrong, is the coinage of *your* brain, not *mine*.) You then put a question and make an assertion:—"*I ask you which of the statements is incorrect, and you persist in trying to change the subject.*" (The former statement is the incorrect one, and your assertion is simply untrue.) You then say:—"*I shall take no notice of any more of your Letters until you tell me explicitly how much of your ground you abandon,—whether you still maintain the side af a xxv.-gon to be* $\frac{1}{8}$ *of the diameter, and its perimeter therefore* $\frac{25}{8}$ *of the diameter—and whether you still maintain the circumference to be* $\frac{25}{8}$ *of the diameter.*" (I have no ground to abandon. I maintain that the circumference of a circle is $\frac{25}{8}$ of the diameter, and the perimeter of a regular inscribed hexagon $\frac{24}{8}$ of the diameter of the circle. As to your taking "*no notice of any more of my letters*," it would probably have been better for your mathematical reputation if you had not forced me into a correspondence, in which case you would not have been troubled with any Letters of mine.)

Paragraph 2. "*I also ask you again, whether you were conscious of Euclid's proof, Book* 3: *Prop.* 35: *Case* 2: *when you said he had not proved, and could not prove, that case.*" (I have read and studied Euclid, and perhaps know as much about every Proposition of Euclid as you do. A living "*recognised Mathematician*" (J. Radford Young) has said:—"*A man may be ignorant of even the multiplication table, and yet be able to master all the Propositions in Euclid.*" You have never heard of W. D. Cooley, and may not have heard of J. Radford Young, but you will find an account of the latter in "*Men of the Time.*") You then say:—"*If you were conscious of it, and were depending on the 'quibble' with which you try to defend yourself, I am sorry for you.*" (I *know* you do not *see* to whom the word "*quibble*" truly applies: but, you may depend upon it, that you stand in much greater need of my commisseration than I do of yours.) You conclude this paragraph by observing:—"*If you were ignorant of it, I would respectfully suggest that you should read Euclid before you publish any more pamphlets on Euclid at Fault.*" (The word respectfully is very prettily introduced into this sentence; and if I should require any of your suggestions, when the time comes for publishing, I may assure you that I will *respectfully* ask for them.)

In one of your early communications you said, *it was your wish that truth should prevail.* I know of no way in which you are so likely to convince me of your sincerity on this point, as by solving the theorems 2 and 3, (I have solved theorem 1 for your "*instruction*") given in my Letter of the 15th instant. This course is still open to you.

Yours faithfully,
JAMES SMITH.

THE REV. PROFESSOR WHITWORTH.

The "*upward path*" of my "*scientific and literary career*" has been beset by *dragons* in various forms, and almost at every step. I gave one of these *dragons*—"*The Round Man in the Square Hole,*"* and he, no less a personage than one of the editorial WE of the *Athenæum*—

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* THE CHRISTMAS CAROL QUADRATURE.

A creed is a very fine thing,
Above all when there is'nt much of it:
There is no π but three-and-an-eighth,
And Mr. James Smith is its prophet.

So here we go round, round, round,
And there we go square, square, square;
Five per cent. is a bob in the pound,
And sixpence an omnibus fare.

When you want to establish a point,
Begin by assuming it true;
For if you are wrong its no matter,
And if you are right it will do.

So here we go, &c.

the opportunity of having "*a bit of sport*" with me, by means of a geometrical figure of which the diagram No. V. is a fac-simile, with certain additions; but he must have *mistaken* it for a "*snake in the grass,*" and consequently shunned it as he would a pestilence! On the 10th December, 1867, I addressed the following Letter to the Editor of the *Athenæum*:—

BARKELEY HOUSE, SEAFORTH,
10th December, 1867.

SIR,

Professor de Morgan said of the *Correspondent:* "*It is one of the most panoramic of the periodicals, there is no knowing*

Propositions ought not to depend
Upon others to keep them in force;
If the creatures can't get their own living,
They must go to the workhouse, of course.
So here we go, &c.

A theorem 's a thing that should jump
Right down its own throat, like a snake;
It can turn itself inside-out thus,
And will do for James Smith to mistake.
So here we go, &c.

May all things come round in the Church;
May all things get square in the State;
But the Union will always shew cracks,
Till 'tis screwed up with three and one-eight.
So here we go, &c.

THE ROUND MAN IN THE SQUARE HOLE.
(Better known as Professor de Morgan.)

These doggrel verses appeared in the *Correspondent* of the 30th December, 1865, a London Journal, which has ceased to exist: and the reader will find that the Editorial WE of the *Athenæum* quotes one of these doggrel verses in that Journal. See "*Our Weekly Gossip,*" *Athenæum*, September 28, 1867. Will Professor de Morgan have the candour to inform the scientific world, how the "*circle squarer in ordinary*" to the *Athenæum* became a contributor to the *Correspondent?* The Professor knows!!! "*Enough on so small a matter.*"

what a week's twopence may bring forth." Well, there is no knowing what a week's *threepence* may bring forth; so, your "*circle-squarer in ordinary*" (Mr. James Smith) "*must take time by the forelock,*" and before he gets his next three-pennyworth, fulfil a promise made in one of his recent Letters, to give you an account of his encounter with a rather remarkable "*dragon*" that beset his path some time ago. Well, then, in the month of February last, I received the following Letter:—

G——, 14*th Feb.*, 1867.

DEAR SIR,

Will you excuse a stranger for intruding upon you? My old and valued friend, Mr. C—— C——, shewed me, a night or two ago, your little book on the "*Quadrature and Rectification of the Circle,*" which greatly interested me; and I am anxious to pay my personal respects to a gentleman who has similar tastes and pursuits with myself. I was in former days a Fellow of Trinity College, Cambridge: and knew the late Dr. Whewell intimately. He often examined me, and I have spent many a hard day on his Statics and Dynamics. I now hold a Trinity living, and came here to settle my youngest son in the office of Messrs. L———.

If you will please to name any time and place, where I may hope to have the pleasure of seeing you, I shall be happy to pay my respects.

I am, dear Sir,
Very faithfully yours,
J—— R——.

JAMES SMITH, ESQ.

I acknowledged the receipt of Mr. R——'s Letter in due course; and, as Mr. C—— C—— was a very old friend of my own, I proposed his house as our place of rendezvous, and fixed the following Saturday afternoon as the time most convenient to myself. Well, we met, and spent the afternoon with our mutual friend—not himself a professed Mathematician, but an excellent arithmetician, and a gentleman of sound common sense—who was greatly amused with "*the things of the day.*"* I went over diagram after diagram, and gave Mr. R—— proof after proof, that a circle and square of equal superficial area, may, and do exist, and can be geometrically constructed. With reference to the geometrical figure represented by the enclosed diagram (*see Diagram V.)*, he said: It appeared to him, as I then explained it, that it settled the long vexed question of "*squaring the circle;*" but he should not like to express a decided opinion, without having the opportunity of studying the diagram carefully; and requested me to favor him with a Letter, giving him, and confining myself to, an explanation of the properties of this particular figure. I thought I had at last fallen in with a Mathematician, whose head was *not* so stuffed "*with crammed erudition, that there was not a cranny hole left for reasoning to get in at;*" and I was only too pleased to comply with his request, and so wrote him the following Letter:—

BARKELEY HOUSE, SEAFORTH,
19*th February*, 1867.

MY DEAR SIR,

"I have much pleasure in complying with the request you made on Saturday, at our very pleasant interview at A—— Hall, to furnish you with a proof that a circle and a square of exactly the same superficial area, may, and do exist, and can be geometrically isolated and exhibited; although this has hitherto been denied by Geometers and Mathematicians. This may be demonstrated in many ways—mathematically as well as geometrically—but I know of no prettier proofs than are afforded by means of the enclosed geometrical figure." *(See Diagram V.)*

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* Mr. M——, an old acquaintance of Mr. C——'s, and an excellent Geometer and Mathematician, accompanied me on this occasion. Of course, Mr. M—— and myself were in complete ignorance of what had passed between the Rev. J—— R—— and Mr. C——, previously to this meeting, but, it was a source of infinite amusement to us, to hear Mr. C—— from time to time exclaim, in great glee:—*Didn't I tell you, I'd find your match?*

"Let A B C be a right-angled triangle of which the sides are in the ratio of 3, 4 and 5 exactly; that is to say, A C : A B : : 5 : 3; A C : B C : : 5 : 4; and, A B : B C : : 3 : 4, by construction. Produce C B to D, making C D = 6 (C B). Raise the perpendicular E D, making E D = A B + B C, and join E C; and thus construct the right-angled triangle E D C. This makes the sides of the triangle E D C in the ratio of 7, 24, and 25 exactly; that is to say, E D : D C : : 7 : 24; E D : E C : : 7 : 25; and, D C : E C :: 24 : 25, by construction. With B as centre and B C as interval, describe the circle X; and with C as centre and C A as interval, describe the circle Y. Produce A B to meet the circumference of the circle X at the point H, and join H C. Produce A C to meet and terminate in the circumference of the circle Y at the point O, and draw G F a diameter of the circle Y, at right angles to A O. Produce C D to L, making D L equal to one-third of D C = 2 (B C). On D L describe the square D L M N, and inscribe the circle Z; and with D as centre and D L as interval, describe the circle X Y Z. With C as centre and C E as interval, describe an arc to meet or cut the line C L at the point K, and draw the chord E K, producing the right-angled triangle E D K. On E K describe the square E K P R."

"The square E K P R and the circles X and Z are exactly equal in superficial area, and my proof of this, is, what I think you particularly requested."

"Now, $\text{Sin}^2 + \text{Cos}^2$ = unity in every right-angled triangle. It is upon this base, or foundation, derived from the 47th proposition of the first book of Euclid, that all the trigonometrical functions or ratios of angles, are raised and established, of which the sine and cosine are the principle, and I am sure you will agree with me in adopting it as a trigonometrical axiom, that $\text{Sin}^2 + \text{Cos}^2$ = unity in every right-angled triangle."

"Now, referring to the Diagram, $\frac{AB}{AC} = \frac{3}{5} = \cdot 6$ = natural sine of angle A C B; and $\frac{BC}{AC} = \frac{4}{5} = \cdot 8$ = natural cosine of angle A C B; and, Sin^2 of angle A C B + Cos^2 of angle A C B = $(\cdot 6^2 + \cdot 8^2) = (\cdot 36 + \cdot 64)$ = unity. But, $\frac{\text{Sin of angle A C B} + \text{Cos of angle A C B}}{5}$

$= \frac{\cdot 6 + \cdot 8}{5} = \frac{1 \cdot 4}{5} = \cdot 28 = \frac{E\,D}{E\,C} = \frac{7}{25}$ = natural Sin of angle ECD; and, 2 (Sin of angle A C D × Cos of angle A C D) = $2 (\cdot 6 \times \cdot 8) = (2 \times \cdot 48) = \cdot 96 = \frac{D\,C}{E\,C} = \frac{24}{25}$ = natural Cos of angle E C D; therefore, Sin^2 of angle E C D + Cos^2 of angle E C D = $\cdot 28^2 + \cdot 96^2 = \cdot 0784 + \cdot 9216 =$ unity; and meets the requirement of the trigonometrical axiom, $\text{Sin}^2 + \text{Cos}^2 =$ unity in every right-angled triangle. But, when A B = 3, and B C = 4, then, A C = 5. Hence: E C = 5 (A C), and E C + D C = 7 (E D), by construction."*

" I shall now direct your attention to a series of facts in connection with the geometrical figure represented by the Diagram, all of which you can readily verify, and I shall not attempt to burden my Letter with the calculations."

" First: If the sum of any two consecutive numbers, and twice their product, represent sides of a right-angled triangle and include the right angle, the triangle is commensurable. Hence: If A B = 3, B C = 4, and A C = 5, then, ABC is what I call the *primary* commensurable right-angled triangle. I so call it, because it is the smallest commensurable right-angled triangle, of which the arithmetical values of the sides can be expressed in whole numbers, and is derived from the consecutive numbers, 1 and 2. In all right-angled triangles derived from two consecutive numbers, the longer of the sides containing the right-angle + 1, is equal to the side subtending the right angle, that is, = hypothenuse."

"Second: From the triangle A B C as the *primary* commensurable right-angled triangle, we may obtain commensurable right-angled triangles, *ad infinitum*. Thus, E D = A B + B C = 7, and D C =

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* In this paragraph I have employed the term natural sine in the same sense as that attached to it by Mathematicians. But, if the sides of a right-angled triangle be 3, 4, and 5: then, ·6 and ·8 are the trigonometrical sines of the acute and obtuse angles, and 3 and 4 the geometrical sines of these angles. Hence, the geometrical sines may vary; or, in other words, we may give the geometrical sines as many arithmetical values as we please; but the trigonometrical sines are invariable, whatever be the arithmetical values of the sides that contain the right angle.

$6(BC)$; or, twice the product of AB and $BC = 24$, therefore, $\sqrt{ED^2 + DC^2} = DC + 1 = 25$, when the arithmetical values of the sides of the triangle EDC are expressed in whole numbers; and these values are derived from the consecutive numbers 3 and 4; that is, from the arithmetical values of AB and BC. I might have added to the diagram the intermediate commensurable right-angled triangle derived from the consecutive numbers 2 and 3, of which the arithmetical values of the sides are 5, 12, and 13."

"Third: The radius of the circle XYZ = the diameter of the circles X and Z, by construction. Hence: $AC + AB = 2(BC) = DE + EN = DK + KL$; and when the radius of the circle XYZ = unity $= 1$. Hence: $\frac{AC}{10} = \left(\frac{BC}{2}\right)^2 = \left(\frac{DE + EN}{4}\right)^2 = \left(\frac{DK + KL}{4}\right)^2$, and all these expressions = area of a square on the semi-radius of the circles X and Z, when the diameter of these circles = unity."

(Compare these facts, Mr. Editor, with the proof I have given you in my Letter of the 5th inst.)

"Fourth: $(AB^2 + BC^2 + AC^2) = (DE^2 + EN^2) = (ED^2 + DK^2) = EK^2 = 12{\cdot}5\left(\frac{DE + EN}{4}\right)^2 = 12{\cdot}5\left(\frac{ED + DK}{4}\right)^2 = 12{\cdot}5\left(\frac{BC}{2}\right)^2 = 50(EN^2) = 50(KD^2)$ and all these expressions $= 3{\cdot}125(BC^2)$."

"Fifth: 6 (radius × semi-radius) = area of a regular inscribed dodecagon to every circle; therefore, $6\left(BC \times \frac{BC}{2}\right) = ED^2 - DK^2 = DE^2 - EN^2$; and all these expressions = area of an inscribed regular dodecagon to the circles X and Z."

"Sixth: $DC = 3(DN)$, by construction; therefore, $6\left(DN \times \frac{DN}{2}\right) = 24\left(BC \times \frac{BC}{2}\right) = DN \times DC$, and all these expressions = area of an inscribed regular dodecagon to the circle XYZ. Hence: The area of the dodecagon is exactly equal to the area of the rectangle $DN \cdot DC$."

"Seventh: $(ED^2 + DK^2 +$ twice the rectangle $(ED \cdot DK) = 4(BC^2) = 16\left(\frac{BC}{2}\right)^2 = 64(DK^2)$, and all these expressions = area of a circumscribing square to the circles X or Z."

"Now, by hypothesis, let D N, a radius of the circle X Y Z, = unity = 1. Then: (E D + D K) = (E D + E N) = D N = 1; and D N = the diameter of the circle Z. But, E D and E N are in the ratio or proportion of 7 to 1; and 3 (D N) = D C, by construction. Hence: $\frac{7}{8}$ (D N) = ·875 = E D : 3 (D N) = 3 = D C: and, $\frac{1}{8}$ (D N) = ·125 = D K; therefore, $\sqrt{E D^2 + D C^2} = \sqrt{\cdot 875^2 + 3^2} = \sqrt{\cdot 765625 + 9} = \sqrt{9 \cdot 765625} = 3 \cdot 125$ = E C, the hypothenuse of the right-angled triangle E D C; and E C = 5 A C, by construction; and it follows, that $\frac{D N}{2} = \frac{1}{2} = \cdot 5$ = B C; and $\frac{3}{4}$ (B C) = A B, by construction; therefore, $\frac{3}{4}$ (B C) = ·375 = A B; therefore, $\sqrt{B C^2 + A B^2} = \sqrt{\cdot 5^2 + \cdot 375^2} = \sqrt{\cdot 25 + \cdot 140625} = \sqrt{\cdot 390625} = \cdot 625$ = A C; therefore, 5 (AC) = 5 × ·62 = 3·125 = E C."

"Again: $E D^2 + D K^2 = \cdot 875^2 + \cdot 125^2 = \cdot 765625 + \cdot 015625 = \cdot 78125 = E K^2$ = area of the square E K P R. But, $\frac{E D + D K}{2}$ = B C, = radius of the circle X, = ·5; therefore, E C (B C^2) = 3·125 (·5^2) = 3·125 × ·25 = ·78125 = area of the square E K P R. But, E C (B C^2) = (A B^2 + B C^2 + A C^2); that is, (3·125 × ·5^2) = (·25 + ·140625 + ·390625) = ·78125; therefore, the sum of the squares of the three sides of the triangle A B C = area of the square E K P R."

"This fixes 3·125 as the true arithmetical value of π, and establishes the truth of the THEORY, that 8 circumferences = 25 diameters in every circle; making the square E K P R exactly equal in superficial area to the circles X or Z; and makes E C, the hypothenuse of the right-angled triangle E D C, exactly equal to the circumference of the circles X or Z."

Proofs:

"I have made D N, the radius of the circle X Y Z, = unity = 1, by hypothesis; and D N = the diameter of the circles X or Z."

"Now, by hypothesis, let E C be greater than 3·125, when D N = 1, say 3·13. Then: $\frac{7}{25}$ (E C) = ·8764 = E D; and, $\frac{1}{25}$ (E C) = ·1252 = D K; therefore, E D + D K = ·8764 + ·1252 = 1·0016, and is greater than unity. This would make the diameter of the circles X and Z greater than unity, which is impossible when D N = 1; therefore, 3·13 must be greater than the true arithmetical value of π."

"Again: By hypothesis, let E C be less than 3·125, say 3·1. Then: $\frac{7}{25}$ (E C) = ·868 = E D; and, $\frac{1}{25}$ (E C) = ·124 = D K; therefore, E D + D K = ·868 + ·124 = ·992, and is less than unity. This would make the diameter of the circles X and Z less than unity, which is impossible, when D N = 1; therefore, 3·1 must be less than the true arithmetical value of π."

"Again: B C, the radius of the circle X, $= \frac{\text{E D} + \text{D K}}{2} =$ ·5, when D N, the radius of the circle X Y Z = 1, and $\frac{5}{4}$ (B C) $= \frac{5 \times {\cdot}5}{4} = \frac{2{\cdot}5}{4} = {\cdot}625 =$ A C. Now, we cannot make the line E C, that is, the hypothenuse of the right-angled triangle E D C, either greater or less than 3·125, without making it either greater or less than 5 (A C), which is impossible, when B C = $\frac{1}{2}$ (D N) = $\frac{1}{2}$ = ·5; and so, every other value of π but $\frac{25}{8}$ = 3·125, '*upsets itself*' (to employ an expression of Professor de Morgan's, in one of his attacks upon me), and demonstrates, beyond the possibility of dispute or cavil, that 8 circumferences equal 25 diameters in every circle."

"Again: I have demonstrated that the area of the square E K P R = the sum of the squares of the three sides of the right-angled triangle A B C, the generating figure of the diagram. Now, the circles X and Z are equal, and their superficial area may be represented by any arithmetical quantity. Take the circle X, and let its area be represented by any finite number, say 75, and be given to find the area of the square E K P R. Then: Since the areas of circles are to each other as the squares of their radii, it follows of necessity, that $\sqrt{\frac{\text{area}}{\pi}}$ = radius in every circle; therefore, $\sqrt{\frac{75}{3{\cdot}125}} = \sqrt{24} =$ B C, the radius of the circle X. $\frac{3}{4}$ (B C) $= \sqrt{13{\cdot}5} =$ A B; and $\frac{5}{4}$ (B C) $= \sqrt{37{\cdot}5} =$ A C; therefore, (B C² + A B² + A C²) $= (\sqrt{24}^2 + \sqrt{13{\cdot}5}^2 + \sqrt{37{\cdot}5}^2) = (24 + 13{\cdot}5 + 37{\cdot}5) = 3{\cdot}125$ (B C²) = 75 = the sum of the squares of the three sides of the triangle A B C, and is equal to the given area of the circle X. But, $\frac{7}{4}$ (B C) $= \sqrt{\frac{7^2}{4^2} \times 24} = \sqrt{\frac{49}{16} \times 24} = \sqrt{3{\cdot}0625 \times 24} = \sqrt{73{\cdot}5} =$ E D; and $\frac{1}{4}$ (B C) $= \sqrt{\frac{1^2}{4^2} \times 24} = \sqrt{{\cdot}0625 \times 24} = \sqrt{1{\cdot}5} =$ D K; therefore, E D² + D K²

$=(\sqrt{73{\cdot}5^2}+\sqrt{1{\cdot}5^2})=(73{\cdot}5+1{\cdot}5)=75=EK^2=$ area of the square E K P R, and is exactly equal to the given area of the circle X. This demonstrates that the circles X and Y are exactly equal in superficial area to the square E K P R; and when D N the radius of the circle X Y Z $=1$, the area of the square E K P R $=12{\cdot}5\left(\frac{\pi}{50}\right)=12{\cdot}5\left(\frac{BC}{2}\right)^2=3{\cdot}125\,(BC^2)=\frac{\pi}{4}={\cdot}78125$; and you may readily convince yourself, that $12{\cdot}5$ times $\frac{\pi}{50}=\frac{\pi}{4}=$ area of a circle of diameter unity, whatever be the value of π; and since the property of one circle is the property of all circles, it follows, that $12\frac{1}{2}$ times the area of a square on the semi-radius = area in every circle.

One proof more: Let E C the hypothenuse of the right-angled triangle E D C be represented by any arithmetical quantity, say $\sqrt{75}$, and be given to find the arithmetical value of the circumferences of the circles X and Z, and the perimeter of a regular inscribed hexagon to these circles. Then: $\frac{1}{25}(EC)=\frac{1}{25}(\sqrt{75})=\sqrt{\frac{1^2}{25^2}\times 75}=\sqrt{\frac{1}{625}\times 75}=\sqrt{{\cdot}0016\times 75}=\sqrt{{\cdot}12}=DK$, the base of the right-angled triangle E D K. $4\,(DK)=4\,(\sqrt{{\cdot}12})=\sqrt{16\times{\cdot}12}=\sqrt{1{\cdot}92}=BC$ the radius of the circle X. But, $2\,\pi$ (radius) = circumference in every circle; therefore, $2\,\pi\,(BC)=6{\cdot}25\,(\sqrt{1{\cdot}92})=\sqrt{6{\cdot}25^2\times 1{\cdot}92)}=\sqrt{39{\cdot}0625\times 1{\cdot}92}=\sqrt{75}=$ circumference of the circle X, and is exactly equal to the given value of the line E C. But, $6\,(BC)=6\,(\sqrt{1{\cdot}92})=\sqrt{6^2\times 1{\cdot}92}=\sqrt{36\times 1{\cdot}92}=\sqrt{69{\cdot}12}=$ the perimeter of a regular inscribed hexagon to the circle X Y Z. Proof: $\frac{3}{\pi}$ expresses the ratio between the perimeter of any regular hexagon and the circumference of its circumscribing circle; therefore, $\frac{\pi}{3}(\sqrt{69{\cdot}12})=\frac{3{\cdot}125}{3}(\sqrt{69{\cdot}12})=\sqrt{\left(\frac{3{\cdot}125^2\times 69{\cdot}12}{3^2}\right)}=\sqrt{\left(\frac{9{\cdot}765625\times 69{\cdot}12}{9}\right)}=\sqrt{\left(\frac{675}{9}\right)}=\sqrt{75}=$ circumference of the circles X or Z, and is exactly equal to the given value of E C. But, I have another proof. $\frac{24}{25}(EC)=DC$; therefore, $\frac{24}{25}(\sqrt{75})=\sqrt{\left(\frac{24^2}{25^2}\times 75\right)}=\sqrt{\left(\frac{576}{625}\times 75=\right)}$

$\sqrt{(\cdot 9216 \times 75)} = \sqrt{69\cdot 12} =$ D C, the base of the right-angled triangle E D C. But: D C = 6(B C), by construction; therefore, the equation, 6(B C) $= \frac{24}{25}$ (E C), is equal to the perimeter of a regular inscribed hexagon to the circles X or Z."

"Now, 4π (E K) = E C^2, that is, 12·5 (E K^2) = E C^2; therefore, $\frac{E\,C^2}{\frac{1}{4}(\pi)} = 16(E\,K)^2$, therefore, 16 (E K^2) = area of a circumscribing square to a circle of which E K is the semi-radius. Of this fact you may readily convince yourself."

Hence:

"The area of a square on E C is equal to the area of a circle, of which E K is the semi-radius."

"E C is exactly equal to the circumference of the circles X or Z."

"D C is exactly equal to the perimeter of a regular inscribed hexagon to the circles X or Z."

"The sum of the squares of the three sides of the right-angled triangle A B C, the generating figure of the diagram, is exactly equal to the area of the square E K P R, and is equal to the area of the circles X or Z."

Hence:

"The area of every circle is equal to the sum of the areas of squares about a right-angled triangle, of which the sides that include the right angle are in the ratio of 3 to 4, and the longer of these sides the radius of the circle."

"Corollary: The area of every circle is equal to the area of a square on the hypothenuse of a right-angled triangle, of which the sides that include the right angle are in the ratio of 7 to 1, and the sum of these two sides equal to the diameter of the circle."

"No other value of π but that which makes 8 circumferences of a circle exactly equal to 25 diameters, is competent to produce such results as I have demonstrated by the geometrical figure represented by the diagram, and establishes beyond the possibility of dispute or cavil, that $\frac{25}{8} = 3\cdot125$ is the true arithmetical value of π; and $\frac{1}{3\cdot125}$ the true arithmetical expression of the ratio of diameter to circumference, in every circle."

"You will observe, my dear Sir, that there are lines in the diagram that I might have omitted, so far as the foregoing proofs are concerned. I introduced them with the idea of going into some proofs by means of angles, but I think I have proved everything you really wished me to prove, and my Letter has already run on to such a length, that I think it better to bring it to a close, and reserve any further observations for some other opportunity."

"Hoping you may find my demonstrations satisfactory, with best wishes for you and yours."

Believe me, my dear Sir,
Very sincerely yours,
THE REV. J—— R——. JAMES SMITH.

Such was my Letter to this Reverend gentleman. And I have never received a reply to, or even an acknowledgment of the receipt of, this Letter ; and you, Mr. Editor, may conceive that I was somewhat surprised when I found that the "*dragon*," *which* had so strangely beset my path, turned out to be rather of the "*winged serpent*" form, than that of a "*dragon*" of the fierce and impetuous bipedal genus. *

In the month of November, 1866, I had written a Letter to my Correspondent, the Rev. Geo. B. Gibbons, embodying all the facts brought out in my Letter to the last "*dragon*" that has beset my path, and this "*dragon*"—as he tells us—knew my Correspondent, the late celebrated "*Master of Trinity*," intimately. That Letter, Mr. Gibbons never condescended to notice in any way whatever; indeed, so far as his opinion is concerned, it might never have been written. Well, then, about the time of my making the acquaintance of the man of *Trinity*, my correspondence with the man of *St. John's* was becoming extremely unsatisfactory.

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* I have only seen the "dragon" of Trinity once, since I sent him the Letter, which I posted with my own hands. On that occasion I reminded him that he had not acknowledged the receipt of it, when he broadly asserted he had never received it. In making this assertion he must have been *romancing*. For, if wrongly directed, or the Rev. J— R— could not be found, the Letter would have been returned to me through the post-office. Verily! Verily! the WE of the *Athenæum* are not the only "*romancers*" to be found in the *Scientific Morality School*. (See the introduction to my pamphlet: *The British Association in Jeopardy, and Professor de Morgan in the Pillory, without hope of escape.*)

In December, 1866, I had written Mr. Gibbons a Letter, and by means of a geometrical figure, represented by a diagram which accompanied it, proved that unequal chords may be subtended by equal arcs.* To this he replied :—"*Yours, just received, asserts that unequal chords may be subtended by equal arcs. So they may, but then the arcs are not arcs of a circle. It is true of an ellipse or parabola, but not true of a circle.*" My answer to this was :—"The assertion contained in this paragraph has amazed me. Referring to the diagram enclosed in my Letter of the 8th of September 1866, the arcs E F D and H G K are equal to one-third part of the circumference of the circles, of which O B is the radius; and the arcs E M H and D N K equal to one-sixth part of the circumference of a circle of which F B = 2 (O B) is the radius. The arcs are therefore equal, and subtend unequal chords. It is true that the arcs are not arcs of the same circle—but when you say '*the arcs are not arcs of a circle,*' I understand you to mean that they are not arcs of any circle. The idea of unequal chords being subtended by equal arcs in the *same* circle, is simply an absurdity. Now, my dear Sir, pray tell me whether you can describe a perfect ellipse, and give me the arithmetical value of its transverse axis? Can you, by means of your compasses, describe a curved line that shall not be the arc of a circle? I can't; and if you can, pray say how?" In a subsequent Letter, I proved, by means of the enclosed diagram, No. 2, *(see diagram VI.)* that "*the arc subtending a side of an inscribed equilateral triangle, to a circle of any radius, is equal to the arc subtending a side of a regular inscribed hexagon to a circle of twice that radius.*" To you, Mr. Editor, this will be obvious, from a mere inspection of the diagram. To Mr. Gibbons it was not obvious, and all he could say, in reply, was :—"*I think it is one of your chief faults, as an investigator of geometrical results, that you employ very complicated figures to perform what (if it can be done at all) can be accomplished by very easy ones.*" My answer was :—"This reminds me of the story of the soldier, who, when being flogged, whether the drummer hit high or hit low, there was no pleasing him. The figure in my last Letter would appear to be—in your opinion—too complicated. Those in my letters of the 19th

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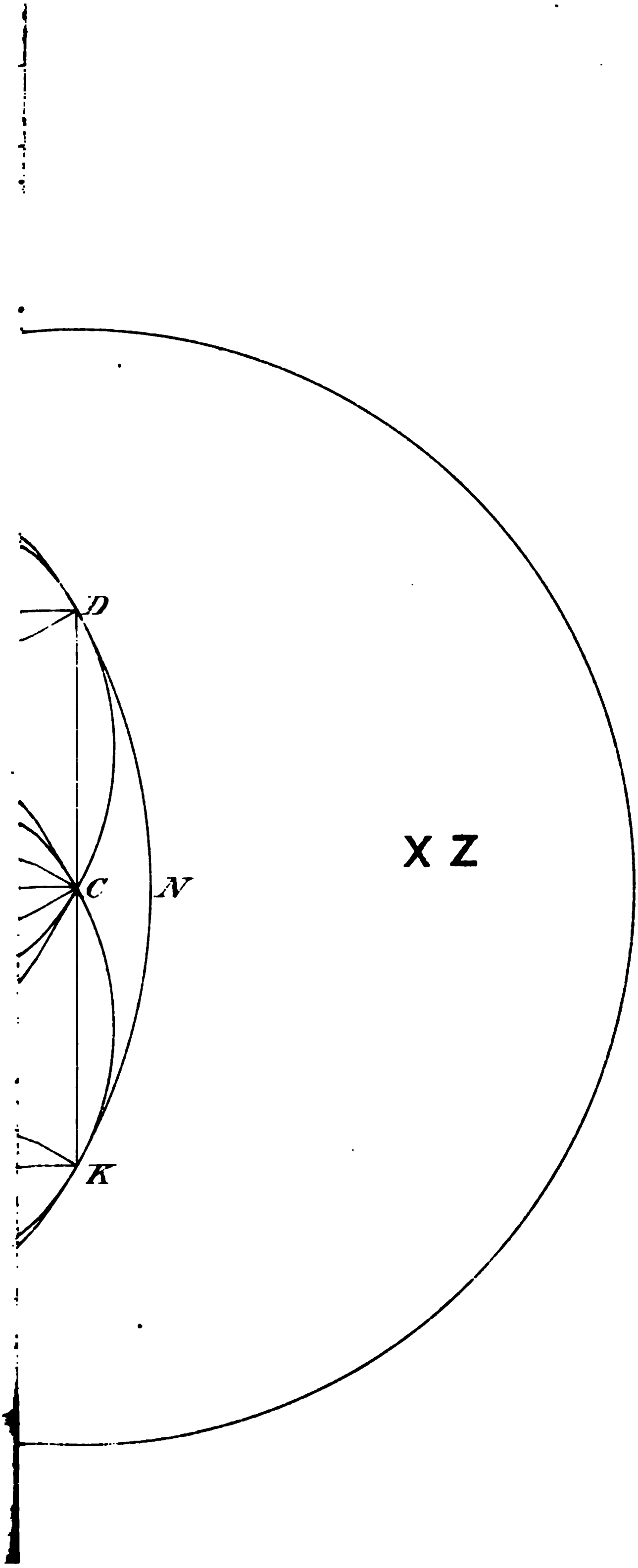
* The diagram here referred to, was very dissimilar to the diagram No. VI., and only contained one circle; but that enclosed in my Letter of the 8th September, 1866, contained the oval or ellipse about the rectangle E H K D, as shewn in the diagram No. VI.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and 28th November, were, I presume, too simple. Be this, however, as it may, one thing is certain, you have never attempted to grapple with the arguments and conclusions derived from any of them."

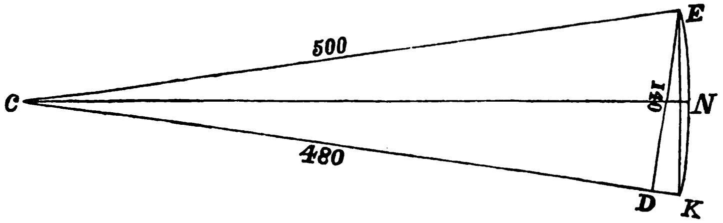
At this period we had given up our dispute about the value of π, and our controversy was on the true solution of a right-angled triangle. This raised another question, the accuracy of our mathematical Tables of sines, cosines, &c. Mr. Gibbons assumed Hutton's Tables to be strictly correct to 7 places of decimals. This I disputed, and gave him some proofs. With my proofs he never attempted to grapple, but met them by the following assertion: "The Tables of sines, cosines, tangents, &c., *have been calculated and printed by experts of every civilized nation in Europe*, and it is unworthy of you to cavil at what is as fixed and certain as the best Interest Tables, and what (*I am sure you would confess*) you have never calculated or investigated at all." In reply, I observed, "This is an inference of your own, and affords another instance of the impetuosity of your natural character, and the consequent rashness with which you jump to conclusions, not only without evidence, but in spite of evidence, aye, and of evidence furnished by yourself;" and I referred him for the proof to a Letter of his own, dated 19th October, 1866, and my reply.

Well, Mr. Gibbons appeared determined to have the last word, and continued to write me. A Letter of mine, dated 28th January, commenced as follows:—"Your Letters of the 18th, 21st, and 23rd inst. are to hand. You commence the first by observing:—'*Your Letter of the 14th inst. shews that you do not understand the construction of the Tables of sines and cosines.*' This is certainly plain speaking, but I venture to tell you, that this is nothing more than a thoughtless *assertion*, and in my opinion, is an assertion that is not very complimentary to yourself as an observer and student of the principles of human nature; for, if true, it follows that you have kept up a twelve months' correspondence with one who is either a knave or a fool, or a remarkable compound of the two. You force me to speak plainly in self-defence, and I hesitate not to tell you what I know to be the fact, that it is you, Sir, who neither understand the construction nor know how to make a right application of the printed tables of sines and cosines, and this I shall prove before I close this Letter." I dealt with each paragraph of Mr. Gibbons's Letters seriatim, and came to one which ran as follows :—

"*Why you fancy (for you make no attempt to compute it) that Sin.* 16° 16′ = ·28 *exactly, I cannot surmise. Calculation shews it otherwise. If H C G be* 16° 16′, *the sides cannot be in the proportion you assign. If the sides are as you fix them H C G is not* 16° 16′ *exactly. In every right-angled triangle, if all the sides are commensurable, the angles will not be expressible in finite terms, and conversely.*"

"Now Sir, I have admitted that ·9899494 is the arithmetical value of the cosine of half the angle H C G (Mr. Gibbons had worked out the proof), and on this point we are agreed. Well then, by referring to the Logarithms of numbers (and we cannot be in *an unhappy state of discord* as to the Logarithms of numbers), we find that the Logarithm corresponding to the natural number ·9899494 is 9.9956130, and on referring to Tables we find that this is the nearest logarithmic cosine to an angle of 8° 8′, and agrees within ·0000035 with the value as given by Hutton. This "*upsets*" your conclusion, that the angle represented by the logarithmic cosine 9·9956095 is an angle of 8° 7′ 49″ nearly ; and this I shall now proceed to demonstrate by pure and simple Geometry, and as I hope to your entire satisfaction."

"Let E D C be a right-angled triangle, of which the sides are represented by 140, 480, and 500. With C as centre and C E as interval describe an arc to meet C D produced at K. Bisect E K at N, and join C N."



You will observe, Mr. Editor, that the triangles E D C and E D K, represent the similar triangles in the enclosed diagram (*See Diagram V.*)

Then : $\frac{E\,D}{E\,C} = \frac{140}{500} = {\cdot}28$ = Sin. of angle E C D ; and, $\frac{C\,D}{E\,C} = \frac{500}{480} = {\cdot}96$ = Cos. of angle E C D, and $E\,D^2 + D\,C^2 = E\,C^2$, therefore, $\sqrt{E\,C^2} = E\,C$ the hypothenuse of the triangle E D C. But, Sin.^{2} of angle E C D + Cos.^{2} of angle E C D $= {\cdot}28^2 + {\cdot}96^2 = {\cdot}0784 + {\cdot}9216$ = unity, and is in harmony with the trigonometrical axiom, Sin.^{2} + Cos.^{2} = unity, in every right-angled triangle. Again : Cosine subtracted from unity = versed Sin. in every right-angled triangle, and C K = C E = unity ; therefore, $C\,K - C\,D = 1 - {\cdot}96 = {\cdot}04 = D\,K$, and D K is the versed sine of the angle E C D. But, E D and D K are sides of the right-angled triangle E D K, and contain the right angle ; therefore, $E\,D^2 + D\,K^2 = {\cdot}28^2 + {\cdot}04^2 = {\cdot}0784 + {\cdot}0016 = {\cdot}08 = E\,K^2$; therefore, $\sqrt{E\,K^2} = \sqrt{{\cdot}08} = {\cdot}28284271$ = E K the hypothenus of the right-angled triangle E D K when E C = unity = 1. But, $\frac{E\,K}{2} = \frac{\sqrt{{\cdot}08}}{2} = \sqrt{{\cdot}02} = {\cdot}14142135$ = N E or N K, and N E or N K is the sine of half the angle E C D. Now, because the angle N K C is common to the two right-angled triangles C N K and D E K, and the angle N C K = half the angle E C D, the angles N C K and D E K are equal. Proof : $E\,K = \sqrt{{\cdot}08} = {\cdot}28284271$, when E C = 1, and D K = ·04, therefore, $\frac{D\,K}{E\,K} = \frac{{\cdot}04}{{\cdot}28284271} = {\cdot}14142135$ = Sin of the angle D E K $= \frac{E\,K}{2} = \frac{\sqrt{{\cdot}08}}{2} = \sqrt{{\cdot}02} = {\cdot}14142135$, and is equal to $\frac{D\,K}{E\,K}$, therefore the angles N C D and D E K are equal.

But further : E D the cosine of the angle D E K = ·28 when E C = 1, therefore, $\frac{E\,D}{E\,K} = \frac{{\cdot}28}{{\cdot}28284271} = {\cdot}9899495$, is the arithmeti-value of the cosine of the angle D E K, and I have proved that D K the sine of the angle D E K = ·14142135, and sin^2 + cos^2 = unity in every right-angled triangle ; therefore, sin^2 of angle D E K + cos.^{2} of angle D E K $= {\cdot}14142135^2 + {\cdot}9899475^2 = {\cdot}0199999982358225 + {\cdot}9800000125502 5 = 1{\cdot}0000000107860725$, and is slightly in excess of unity. This arises from the very obvious fact, that in dividing the value of E D by the value of E K, the divisor is slightly less than its true value, and necessarily makes the quotient slightly in excess of its true value. But, *as you have shewn*, we have another method of finding the cosine of half the angle E C D, that is, the cosine of the angle N C D, by which we make it ·9899494 (and on this point we are in a state of "happy concord"), therefore, $\sin^2$ of angle N C D + $\cos.^2$ of angle N C D = $·14142135^2 + ·9899494^2 = ·0199999982358225 + ·9799998145636 = ·9999998127961825$, and is slightly less than unity.

Thus, I make the sine of the angles N C K and D E K = ·14142135 and the cosine of these angles ·9899494 or ·9899495, and I care not which we adopt, the only difference being that one makes $\text{Sin}^2 + \text{Cos}^2$ slightly less and the other slightly greater than unity, but either meets the requirement of the trigonometrical axiom as nearly as it is possible to do so by means of logarithms, and demonstrates that Hutton's Tables are in error in making the sine and cosine of an angle of 8° 8′ to be ·1414772 and ·9899415. There is a fallacy in the method by which experts have been taught to make the calculations of sines and cosines for our mathematical tables, and yet, I may frankly admit, that limited in there application to degrees and minutes, they are sufficiently near the truth to be available for many practical purposes. But, Sir, when you come to make your calculations by means of them to seconds, and even parts of a second, as you have to do in some of your astronomical calculations, they lead you into error, and into "*confusion worse confounded.*" You are aware that I have propounded a theory with reference to the relations existing between the dimensions and distances of the heavenly bodies, and I venture to tell you, that the day is not far distant (it may not be in my time) when that theory will become an admitted fact.

In my Letter to you, Mr. Editor, of the 28th October, I have given the concluding paragraphs of my Letter to Mr. Gibbons of the 28th January, to which I may refer you.

Such was the very unsatisfactory state of the controversy between me and the "*dragon*" of St. John's when the "*dragon*' of Trinity beset my path. I had lost all hope of finding a *cranny* in the head of the former, at which *geometrical reasoning* could get in. Of the latter I had formed a favourable opinion, and hoped to have found in him a champion fighting by my side in the cause of scientific truth ; and I must confess I was surprised, and I may say disappointed, at the course pursued by him, a course so thoroughly inconsistent with "*the usual courtecies of good society.*" Hence, I am still left alone, to fight the battle of cyclometry with the "*dragons*" of the mathematical world.

If I am not driven from my course by my next three pennyworth, I shall take a fresh departure from the point where my Letter of the 4th November left us, when I next address you.

I am, Sir,
Yours respectfully,
JAMES SMITH.

Mr. Gibbons had probably destroyed, or at any rate, must have forgotten my Letter of the 8th September, 1866, when he penned the following sentence: "*Yours just received, asserts, that unequal chords may be subtended by equal arcs. So they may, but then the arcs are not arcs of a circle. It is true of an ellipse or parabola, but not of a circle.*" In a Letter to Mr. Gibbons, dated the 22nd December, 1866, I dealt with this assertion, by means of the geometrical figure represented by diagram No. VI., and by very simple, but logical reasoning, worked out the following conclusion:—"*Hence: The arc subtending a side of an inscribed equilateral triangle, to a circle of any radius, is equal to the arc subtending a side of a regular inscribed hexagon to a circle of twice that radius.*" To you, Sir, this will be self-evident from a mere inspection of the diagram; and it follows, that the periphery of the oval or ellipse about the rectangle E H K D, is equal to two-thirds of the circumference of the circles X Y or X Z; and four-thirds of the circumference of the circles X or Y.

In December, 1867, I received a Letter from the Mr.

S—— referred to in the early part of my pamphlet "*Euclid at Fault*," enclosing one from his brother, in which he went into what he thought proofs of the fallacy that $\pi = \frac{25}{8} = 3{\cdot}125$. My Letter in reply, dated 17th Dec., 1867, commenced as follows:—

> "I am in receipt of your esteemed favour of the 13th inst., and beg to thank you for the interest you have taken in my labours on "The *Quadrature and Rectification of the Circle*."
>
> "With your brother's calculations, enclosed in your Letter, I was quite familiar. The error involved in them, is not one of calculation, but of principle. The 47th Proposition of the 1st Book of Euclid treats of a rectilinear figure, and is therefore inapplicable *directly* to measure a curvilinear figure, but *indirectly* it can be made available in many ways to prove the ratio of diameter to circumference in a circle."
>
> "I can prove the true ratio by a comparison of a curvilinear line with curvilinear lines, in a way so plain and simple, that I think your own knowledge of Geometry and Mathematics will be quite sufficient to convince you of its truth."

I enclosed a copy of the geometrical figure represented by the diagram No. VI., and from it worked out an algebraical formula, which solves the ratio of diameter to circumference in every circle. Mr. S—— forwarded my Letter to his brother, who handed it to his relative, Mr. R——, and on the 1st, January, 1868, I recived a Letter from Mr. S——, enclosing Mr. R——'s observations on my Letter. The following is a literal quotation from Mr. R——'s Paper:—

> "Remarks on the alleged proof that $\pi = 3\frac{1}{8}$."
>
> "Referring to pages 3 and 4 of the paper sent to Mr. S——, A = area of square circumscribing a circle, to find the area of a circle and value of π. m and n = circumference of larger and smaller circles: p = periphery of the figure about the rectangle. Now, the only ***known*** quantity here is a, which being the area of circumscribing square is $= d^2$, d being the diameter. And therefore it is impossible

to find, from this, the area of the circle and π, without introducing some mathematical principles or principle applicable to the case, and reasoning from it or them. One principle, or fact, is put as a premiss, viz.: that 16 times square of half-radius of a circle = the circumscribing square, which of course is self-evident. Another fact put as premiss or datum, is $\frac{2}{3}(m) = \frac{4}{3}(n)$, which is also true. The circumferences being as the diameters, this quantity $\frac{4}{3}(n) = \frac{2}{3}(m) = p$, is also true. But the figure is called an ELLIPSE. If it is meant, that it is an *ellipse proper*, one of the conic sections, this is a mistake. It cannot be, and is not an ellipse."

Well then, here we have Mr. Gibbons and Mr. R——, both "*recognised Mathematicians,*" * differing from each other as to the properties of an ellipse. The former admitting that the oval figure about the rectangle E H K D may be an ellipse or parabola; and the latter broadly maintaining that it is not, and cannot be an "*ellipse proper.*" To you, Sir, it will be—or at any rate ought to be—self-evident, that it is the only perfect *ellipse* that can be constructed by rule and compasses, and that the transverse axis of it is equal to three times the radius of the circles X or Z.

JAMES SMITH *to* THE REV. PROFESSOR WHITWORTH.

BARKELEY HOUSE, SEAFORTH,
22nd December, 1868.

SIR,

In my Letter of the 19th instant, I have proved, in several ways, by sound reasoning from indisputable data, that the true arithmetical value of π is $\frac{25}{8} = 3{\cdot}125$. My demonstrations will be readily understood by, *and convincing to*, a first-class schoolboy; and no Mathematician in the world can controvert them. I have no *doubt* that you *know* this; but I have no *idea* you will ever admit it, notwithstanding your professions of a "*desire that truth should prevail.*" I could furnish many other proofs by constructive Geometry, but this will be quite unnecessary to any "*reasoning geometrical investigator.*" The enclosed diagram (*see Diagram VII.)* must be taken, in connection with the geometrical figure represented by the diagram enclosed in my Letter of the 15th instant.

---

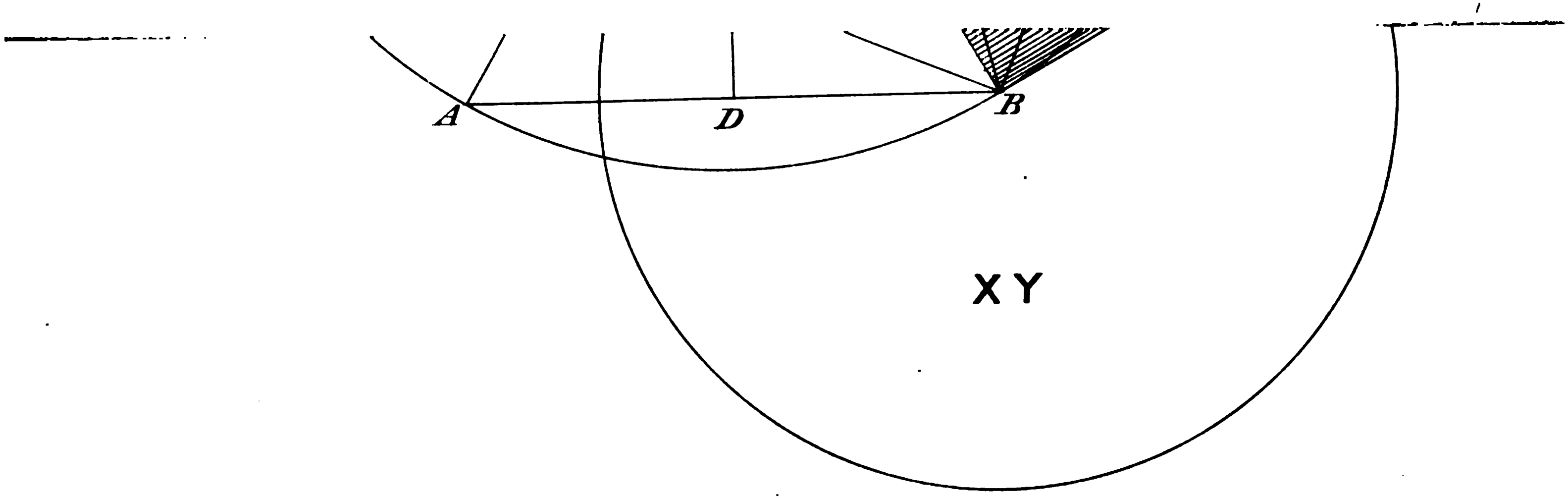
* The Rev. Geo. B. Gibbons is the friend and intimate of Professor Adams, and in one of his Letters informed me, that he had calculated the times of eclipses for a scientific work, going back for upwards of 2000 years. I am not at liberty to give the name of Mr. R——, and, consequently, cannot furnish the proof.

CONSTRUCTION.

Let A and B denote points dotted at random. Join A B, and on A B describe the equilateral triangle O A B. Draw the line O D, bisecting the angle O and its opposite side A B. With O as centre and O A or O B as interval, describe the circle X, and produce B O to meet the circumference of the circle X at the point K. With B as centre and B F equal to $\frac{3}{8}$ (B K) as interval, describe the circle X Y. From the point B draw a straight line at right angles to K B, and therefore tangental to the circle X, to meet and terminate in the circumference of the circle X Y, at the point T, and join O T, and thus construct the right-angled triangle O B T. Produce B T to the point P, making T P equal to $\frac{7}{24}$ (B T). From the point F draw a straight line perpendicular to K B, and parallel to B P, and therefore tangental to the circle X Y, to meet another straight line drawn from the point P, parallel to B F. These lines meet in the circumference of the circle X, at the point H, producing the rectangular parallelogram F B P H. Draw the diagonals F P and H B, and join K H and H T. From the point B, draw a straight line through the point of intersection, between the circumference of the circle X Y and O T the hypothenuse of the right-angled triangle O B T, to meet and terminate in the line F H, at the point V. Or, from F H cut off a part, V H, equal to B T, and join V B. The result is the same. From the point H, draw a straight line through the point O, the centre of the circle X, to meet and terminate in the circumference of the circle at the point L, and join K L and B L, and thus construct the inscribed rectangular parallelogram K L B H to the circle X.

Now, K H B is a right-angled triangle. (Euclid: Prop 31 :

DIAGRAM VII.



Book 3). On this point we are agreed. But, because H F is perpendicular to K B, by construction, H F O is a right-angled triangle: and H O = O B, for they are radii of the circle X. B T = B F for they are radii of the circle Y, and V H = B T, for they are opposite sides of a parallelogram: and H P = B F, for they are also opposite sides of a parallelogram. But, B T F is a quadrant of the circle X Y, and it follows, that the angles H B T, H B V, and V B F are together equal to a right angle. But, I have proved in a previous Letter that the angles B O T and T H P are together equal to the obtuse angle O T B, in the right-angled triangle O B T; and that the angles B O T, O T B, T H P, and H T P are together equal to two right angles. But, the triangles B F V and H P T are similar and equal right-angled triangles, by construction; and it follows, that the angles B O T, O B V, F V B, and O T B, are together equal to two right angles. Now, the angles H B T and H B V are angles of 36° 52′, and the angle V B F an angle of 16° 16′, and the three angles are together equal to the right angle F B T = 90°. I have, in my last Letter, directed your attention to the fact, that with reference to the diagram enclosed in my Letter of the 15th inst, the angles A C B, and B C H are angles of 36° 52′, and the angle H C G an angle of 16° 16′, and the three angles together equal to the right angle A C G; and A H G C is a quadrant of the circle Y. But, you will observe that the angle H C G is an angle contained by two radii of the circle Y, while, in the enclosed diagram, the equal angle is the acute angle of the right-angled triangle B F V. If you are unable to convince yourself of these facts, "*I can't help it, but the facts remain notwithstanding.*"

This leads me to the especial object for which I bring the geometrical figure, represented by the enclosed diagram *(see Diagram VII)* under your notice.

Let K B the diameter of the circle X, be represented by the number 8. Then: O B and O H = 4; F B = 3; and O F = 1, by construction, and we must find the values of other lines in the figure; by computation.

Then: By Euclid: Prop. 47: Book 1.

$(HO^2 - OF^2) = (4^2 - 1^2) = (16 - 1) = 15 = HF^2$; therefore, $HF = \sqrt{15}$. But, K F H is a right-angled triangle; therefore,

$KF^2 + FH^2 = (5^2 + \sqrt{15}^2) = (25 + 15) = 40 = KH^2$, therefore, $KH = \sqrt{40}$.

By Euclid : Prop. 12 : Book 2.

$\{HO^2 + OK^2 + 2(OK \times OF)\} = KH^2$: that is, $\{4^2 + 4^2 + 2(4 \times 1)\} = (16 + 16 + 8) = 40 = KH^2$ therefore, $KH = \sqrt{40}$.

By Euclid : Prop. 8 : Book 6.

$(KB \times KF) = (8 \times 5) = 40 = KH^2$; therefore, $KH = \sqrt{40}$.

Now, $HP = BF$, for they are opposite sides of the parallelogram F B P H, and $BT = BF$, for they are radii of the circle X Y; therefore, $BT = 3$, when the diameter of the circle $= 8$. But, $TP = \frac{7}{24}(BT)$, by construction; therefore, $\frac{3 \times 7}{24} = \frac{21}{24} = \cdot875 = TP$, and H P T is a right-angled triangle; therefore, $HP^2 + TP^2) = (3^2 + \cdot875^2) = 9 + \cdot765625 = 9\cdot765625 = HT^2$; therefore, $\sqrt{9\cdot765625} = 3\cdot125 = HT$; therefore, $HT = \pi$; that is, H T = circumference of a circle of diameter unity.

Now, H P B is a right-angled triangle, and H T B, a part of it, is an oblique-angled triangle, and by Euclid: Prop. 12 : Book 2 : $\{HT^2 + TB^2 + 2(TB \times TP)\} = HB^2$; that is, $\{3\cdot125^2 + 3^2 + 2(3 \times \cdot875)\} = HB^2$; or, $(9\cdot765625 + 9 + 5\cdot25) = 24\cdot015625 = HB^2$. But, H P B and H F B are similar and equal right-angled triangles, and by Euclid : Prop. 8 : Book 6 : and Prop. 35 : Book 3 : $HF = \sqrt{15}$, when $KF = 5$, and $FB = 3$; and $FB = 3$, by construction, when the diameter of the circle $X = 8$; therefore, $(HF^2 + FB^2) = (\sqrt{15}^2 + 3^2) = (15 + 9) = 24 = HB^2$. How is this? Can the line H B have different arithmetical values when the diameter of the circle $X = 8$? Can the diagonals of the rectangular parallelogram have different arithmetical values under any circumstances? Well, then, *is not Euclid at fault?*

Again : The sum of the squares of the four sides, is equal to the sum of the squares of the diagonals, in every parallelogram. Hence: $(FB^2 + BP^2 + FH^2 + HP^2) = (FP^2 + HB^2)$; that is, $(3^2 + 3\cdot875^2 + 3\cdot875^2 + 3^2) = (24\cdot015625 + 24\cdot015625)$, or, $(9 + 15\cdot015625 + 15\cdot015625 + 9) = 48\cdot03125$. If you were to tell me—as probably you will, if you write me again—that $BP = FH = \sqrt{15}$, when the diameter of the circle $X = 8$; consequently,

$\{2(15 + 9\} = 2(24) = 48$, and so insinuate that you had got over the difficulty; this would be what I should call a "*quibble.*" You may shirk the fact, but you cannot controvert it, that T P is equal to $\frac{7}{96}$(B T), by construction: and it follows of necessity, that *Euclid is at fault.* The area of the parallelogram F B P H is $3 \times 3·875 = 11·625$. You would make it $3(\sqrt{15}) = \sqrt{3^2 \times 15} = \sqrt{9 \times 15} = \sqrt{135} = 11·618$, &c. How can this be? Can the arithmetical value of the area be different in the same parallelogram? Now, that Euclid is at fault, could only be discovered, and can only be demonstrated, by "*wielding that indispensable instrument of science, Arithmetic.*" Well, then, it just comes to this—as I told you in my Letter of the 11th instant—that when the $\sqrt{15}$ represents a line, it stands, not for the arithmetical quantity 3·872, &c., its extracted root, but for 3·875.

Again: K H B and B L K are similar and equal right-angled triangles, and K B the diameter of the circle X, is the hypothenuse of, and common to, the two triangles. Take the triangle K H B. Then: $(KB^2 — BH^2) = (8^2 — \sqrt{24·015625}^2 = (64 — 24·015625) = 39·984375 = KH^2$; therefore, $KH = \sqrt{39·984375}$. Now, 39·984375 is the true area of a square on K H, and 24·015625 the true area of a square on H B, when the diameter of the circle $X = 8$; the former less than 40 and the latter greater than 24. But, these differences are compensating. Hence: whether we call KH^2 and HB^2, $\sqrt{40}$ and $\sqrt{24}$, or, $\sqrt{39·984375}$ and $\sqrt{24·015625}$, the sum of the squares of the four sides of the parallelogram K L B H is equal to the sum of the squares of the diagonals, and both are equal to twice the area of a circumscribing square to the circle X: and this is true of every rectangular parallelogram inscribed in a circle.

Your Letter of the 15th December commences thus: "*Your words* (December 8th) *are:—The square of a line drawn from the circumference of a circle perpendicular to its diameter is equal to the rectangle under the segments of the diameter. Euclid nowhere proves this, nor could he, by pure Geometry apparently.*" Why did you not quote the paragraph in its entirety? After the word geometry I added, *but apparently, it is readily demonstrated by applied mathematics.* In this sort of way you have repeatedly perverted what I have said. The next paragraph of that Letter runs thus :—"*I have shewn you that he does prove it explicitly and absolutely, and you now say he does not prove it mathematically. You must give a strange definition to the word mathematics. Please say what your definition is, that I may understand you for the future. You appear to mean that Euclid does not express the perpendicular and the segments arithmetically. It is well that he does not, for if he did his proof like yours would only apply to a particular case. But he proves absolutely that the square and rectangle in your enunciation are equal geometrically and absolutely, and therefore they must be equal whatever be the common unit of area in which any one may be pleased to express them.*" Now, Sir, if you are right in what you say in this quotation, you can readily prove me to be wrong in making H B = 24·015625, when K F = 5 ; F B = 3 ; and T P = $\frac{7}{24}$ (F B) or $\frac{7}{24}$ (B T); and if I am right, a square *on* the line F H is greater than 15, and therefore greater than K F × F B, and it follows, that *Euclid is at Fault* in the 8th proposition of his sixth book, and 35th proposition of his third book.

In my Letter of the 19th inst., I directed your attention to the fact, that if the radius of a circle be represented by the arithmetical symbol 5, the difference between the area of the circle and its inscribed square = $\frac{360}{10}\left(\frac{\pi}{4}\right)$ = 28·125. Now, I have proved that H T = 3·125 = π, when T P = $\frac{7}{24}$ (B T), and B T = H P. But, F B = H P, and V B = H T, and it follows, that V B is equal to the circumference of a circle of diameter unity, and F B equal to the perimeter of a regular inscribed hexagon to a circle of diameter unity. Hence: F B^2 × V B = area of the circle X Y, when the diameter of the circle = 8 ; that is, $3^2 \times \pi = 9 \times 3·125 = 28·125$: and it follows, that if with F as centre and F K as interval we describe a circle, and inscribe a square, the difference between the area of the circle and the area of the inscribed square will be exactly equal to the area of the circle X Y. But the area of a circumscribing square to a circle of which F K is the radius = 100 when the diameter of the circle X = 8, and it follows, that the area of an inscribed square = $\frac{100}{2}$ = 50. $\div$ $3\frac{1}{8}$ (O B^2) = area of the circle X. The sum of the areas of the circles X and X Y is equal to the area of a circle of which F K is the radius, since F K = O T; or in other words, since the radii of the three circles is equal to the sides of the right-angled triangle O B T: and it follows, that if a square be inscribed in a circle of which F K or O T is the radius, this square and the circle X will be exactly equal in superficial area.

Now, Sir, I admit that you are a "*recognised Mathematician*," and, *as such*, it can cost you but little time or trouble to try to work out these results with the value of π as assigned by Mathematicians, or by any other hypothetical value of π. If you find all other values of π fail but $\frac{25}{8} = 3 \cdot 125$, will not this convince you—if you wish to be convinced—that 3·125 must be the true arithmetical value of π; and that the problem of squaring the circle is "*un fait accompli?*" If you are determined not to be convinced, or at any rate not to admit it, how can you expect me to believe your profession, that *your only desire is that TRUTH should prevail?* If you believe in the maxim, "*Do unto others as you would have others do unto you*," you will take the earliest opportunity of withdrawing the *foul* charges you brought against me in your original communication.

Faithfully yours,

JAMES SMITH.

THE REV. PROFESSOR WHITWORTH.

THE REV. PROFESSOR WHITWORTH *to* JAMES SMITH.

LIVERPOOL, *December 28th*, 1868.

MY DEAR SIR,

In my last communication I told you that I would not pursue a correspondence with you while you persisted in shifting your ground. I informed you that I would attend to no more Letters until you chose to attend to my strictures on your earlier proofs; either abandoning your arguments, or answering my objections to them. I shewed you that it followed, from your position, that the perimeter of a xxv.-gon, and the circumference of its circumscribing circle, were each $\frac{25}{8}$ths of the diameter of the circle, and therefore equal. I declined to give you any more attention, till you should tell me whether you maintained this equality; or, if not, whether you withdraw your measurement of the polygon, or of the circle. At length I have received a Letter, dated 19th December, in which you deny that anything you have written is equivalent to the assertion that the perimeter of a xxv.-gon is $\frac{25}{8}$ths of the diameter. But if you refer again to your Letter of the 23rd November, 1868, in which you have drawn the triangle A B C one of 25 equal isosceles triangles inscribed in a circle [with A as centre, and A B or A C as radius], you will see that you have stated that the natural sine of angle B A D [A D being perpendicular to B C] is ·125, or $\frac{1}{8}$th.

Now, the sine means the ratio of the perpendicular B D to the radius A B: therefore, on your shewing, B D is $\frac{1}{8}$th of B A, and therefore B C (the double of B D) is $\frac{1}{8}$th of the diameter (the double of B A). But the perimeter of the xxv.-gon is 25 times B C, and therefore $\frac{25}{8}$ths of the diameter, and this you give also as the circumference of the circle.

Now, I simply ask you to point out, if you can, any false step in the reasoning of the last ten lines. If you cannot shew a flaw in the reasoning (and I know that you cannot), it follows from your statements that the perimeter of the xxv.-gon is equal to the circumference of the circumscribing circle; and, is not the arc equal to the chord?

I have not had time to write this sooner, as my time has been entirely taken up with more important matters.

I have not yet read more than the opening paragraph of your last Letter. In that paragraph you say you have no doubt that I know that $\pi = \frac{25}{8}$, though you are sure I will not admit it. In other words, you imply that I am *consciously lying*, when I tell you your arguments are unsound and your conclusions fallacious, and that I am, in fact, knowingly bolstering up a cause in which I do not believe. Were I guilty of this, I should resign all claim to be a gentleman and a man of honour.

I expect that, in your next Letter, you will apologise for this scurrilous paragraph. If you do not, I shall return, unopened, any subsequent communications that you may address to me.

I remain, in expectation of your withdrawal of the paragraph in question,

Faithfully yours,
W. ALLEN WHITWORTH.

There were several ways open to me of shewing the fallacy of the reasoning in the 10 lines to which Professor Whitworth especially refers. My correspondent, the Rev. George B. Gibbons, assumed that because Logarithmic Tables "*have been calculated by experts in every country in Europe,*" they must necessarily be correct. To prove that the arithmetical value of π must be greater than 3·125, (referring to an isosceles triangle, A B C, similar to that in my Letter of the 23rd Nov., 1868,) he put his argument in the following way:—

$$\text{The angle } B\,A\,C = \frac{360^\circ}{25} = 14^\circ\,24'.$$

$$\text{The angle } D\,A\,B = \frac{14^\circ\,24'}{2} = 7^\circ\,12'.$$

Therefore, B D = half the chord B C = B A sine of $7^\circ\,12'$ = sine of $7^\circ\,12'$ for B A radius = 1. The natural sine of $7^\circ\,12'$, as per Hutton's Tables, is ·1253333: therefore, 2 (·1253333) = ·2506666 = the chord B C, and is greater than $\frac{3\cdot125}{12\cdot5} = \cdot25$. Hence: Mr. Gibbons drew the conclusion, that this would make the chord B C greater than its subtending arc.

There is the same fallacy in this reasoning, as in Professor Whitworth's; indeed, the argument is precisely similar when carefully analyzed. It is based on the *implied assumption* that the trigonometrical functions of angles are lengths, and not ratios. Now, assuming the argument to be sound, it follows of necessity, that the natural sine of an angle of $14^\circ\,24'$ is the double of ·1253333, or ·2506666: but on reference to Hutton's Tables we find, that Hutton makes the natural sine of this angle to be only ·2486899, which is less than $\frac{3\cdot125}{12\cdot5}$ = ·25. Now, either Hutton is wrong, or the argument of Mr. Gibbons and Professor Whitworth unsound. Both cannot stand the test of examination, "*by the rules of logic and common sense.*" The fact is, $\frac{\pi}{12\cdot5}$ is equal to 100th part of the area of a circle of diameter 8, whatever be the value of π, but it requires a Mathematician to have "*the two eyes of exact science*" open to see it.

Well, then, it appears to me that if Professor Whitworth was not insincere in the use he has made of the *lapsus* in the paragraph on page 10 of my Letter of the 23rd November, he must have believed me to be "*consciously lying,*" in my reiterated denials that I make a certain chord equal to its subtending arc. Horace Walpole, in one of his Letters to Sir Horace Mann, observes:—"*This world is a comedy to those who think, a tragedy to those who feel.*" It has been well said, "*Oh, that mine enemy would write a book.*"

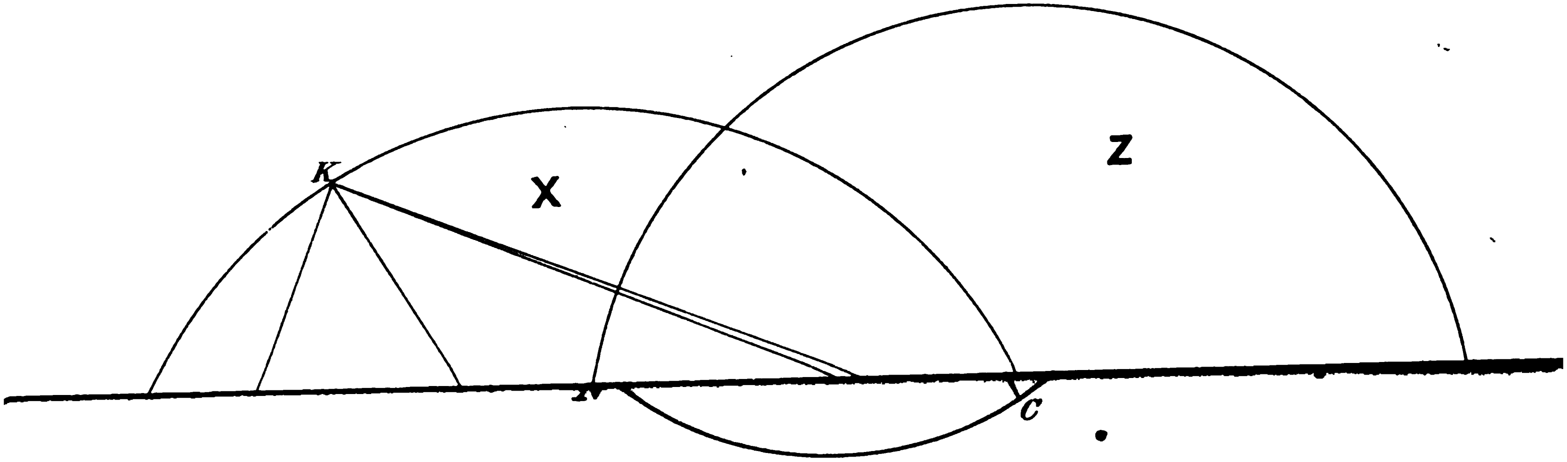
JAMES SMITH *to* THE REV. PROFESSOR WHITWORTH.

BARKELEY HOUSE, SEAFORTH,
29th December, 1868.

SIR,

In your Letter of the 13th November you observed:—"*I must decline for the present to sanction your wish to lay my private Letters before the third person to whom you refer in such extraordinary terms.*" The gentleman here alluded to is Mr. James M. Wilson, Fellow of St. John's College, Cambridge, senior wrangler in 1859, and now Mathematical Master of Rugby School; and if you *prove* that I have done either you or him an injustice, I shall be prepared to make the most ample apology. You are no doubt aware that Mr. Wilson has recently published a treatise on "*Elementary Geometry,*" as a better text-book for teaching the rudiments of that

DIAGRAM VIII.



science than Euclid's elements. It was reviewed—not favorably—in the *Athenæum* of the 18th July last. That review is obviously from the pen of De Morgan, and I quote the following from it:—"*What direction is we are not told, except that 'straight lines which meet have different directions.' Is a direction a magnitude? Is one direction greater than another? We should suppose so; for an angle, a magnitude, a thing which is to be halved and quartered, is the difference of the direction of two straight lines that meet one another. A better definition follows; the* 'QUANTITY OF TURNING' by which we pass from one direction to another. But hardly any use is made of this, and none at the commencement." Mr. Wilson has made his meaning of the expression "*quantity of turning*" plain enough in his sixth definition, and I have no doubt you know as well as I do, that it is hardly possible to apply this expression in teaching the problems and theorems of Euclid; but the term "*quantity of turning*" becomes of the utmost importance when we come to apply what we have learned by pure geometry, to practical purposes. De Morgan, in his capacity of Mathematical critic to the *Athenæum*, has said: "*We hope to have many a bit of sport with him* (Mr. Smith) *in the future, as we have had in the past*," and I have no doubt that if *I* had made use of the expression "*quantity of turning*," he would have had a rare bit of sport with me; but having been introduced by a "*recognised Mathematician*" of such celebrity as Mr. J. M. Wilson, I presume I may now be excused for making use of the term, and turning it to practical account. Well, then, I shall now direct your attention to some extraordinary consequences that result from a certain "*quantity of turning*," in practical or constructive geometry.

The geometrical figure represented by the enclosed diagram *(see Diagram VIII.)*, is a fac-simile of that in my Letter of the 22nd inst., with the following additions:—From the line H F cut off a part m F, making m F equal to $\frac{3}{4}$ (K F), and join K m and m B. With m as centre and m F as interval, describe the circle Z, and produce K m to meet and terminate in the circumference of the circle Z, at the point n; and join B n. Produce K n to meet B P produced, at the point M. Produce O D and O B to meet the circumference of the circle X Y, at the points N and C.

In his sixth definition Mr. Wilson observes: "*Two straight lines that meet one another have different directions, and the differ-*

ence of their direction is the angle between them." Thus, the direction of O N in the enclosed diagram, is different from the direction O C, and the difference of their direction is the angle N O C. In the same definition Mr. Wilson says: "*An angle may be conceived as generated by the revolution of a line A B, starting from some initial position A C, and the angle is the* QUANTITY OF TURNING *required to make A C coincide with A B.*"

Now, referring to the enclosed diagram, conceive the line O N to revolve in the direction of B, "*starting from*" the "*initial position*" O D, until it coincides with O C, the difference of their direction is the angle N O C, which is an angle of 30°. Again: Conceive the line O C to revolve towards P, "*starting from*" the "*initial position*" O B, until it coincides with O T, the difference of their direction is the angle C O T, which is an angle of 36° 52′. Again: Conceive the line K C to revolve towards M, "*starting from*" the "*initial position*" K B, until it is parallel with O T, it will then coincide with K M, and the difference of the direction of K C and K M is the angle B K M, and is a similiar angle to the angle B O T; or in other words, the triangles O B T and K B M are similiar right-angled triangles, and the sides that contain the right angle are in the ratio of 3 to 4. It is self-evident that a further "*quantity of turning*" must be applied to the line K C to make it coincide with the line K H, a side of the parallelogram K L B H inscribed in the circle X.

K B and L H are diameters of the circle X. Now, we can conceive a "*quantity of turning*" applied to K B in the direction from B to D, until it should coincide with O D: and we can conceive a "*quantity of turning*" applied to L H in the direction from H towards B, until it should become parallel to A B, the generating line of the diagram. In this case K B and L H would become diameters of the circle X at right angles to one another: and you will observe that the "*quantity of turning*" applied to both, is in the same direction. But, we can conceive the line K C and the diameter L H to revolve in opposite directions; that is to say, the former in the direction from B towards M, and the latter in the direction from H towards T, until they coincide, when K C would cut the line B M at a point intermediate between T and P, but would not be parallel to O T and K M, but parallel to K H and L B, You will observe, that K B, a part of K C, is a diameter of the circle X. Now, we can conceive the diameters of the circles to revolve *simultaneously* in both cases, and we can also conceive that they might be the diagonals of rectangular parallelograms at every stage of their revolutions. In the former case, the rectangular parallelograms would enclose a larger portion of the area of the circle X at every stage in the revolution of the diameters, and finally, become an inscribed square to the circle, which encloses a larger portion of the area of the circle, than any other form of inscribed rectangle. In the latter case, the rectangles would enclose a smaller portion of the area of the circle, at every stage in the revolutions of the diameters; and finally, the diameters of the circle would reach the point of coincidence, and no longer denote the diagonals of a rectangle inscribed in the circle.

With my Letter of the 28th November, I sent you a copy of my Pamphlet "*Euclid at Fault*," in which there is a diagram shewing a rectangular parallelogram, K L B H, inscribed in a circle. In my Letter of the 23rd November there was a diagram enclosed *(see Diagram I.)*, also shewing a rectangular parallelogram, C K A D, inscribed in a circle. In reply to the former Letter, you favoured me with the lengths and breadths of 17 rectangles. I did not expect this. I thought you *might* give me the sides of the rectangles in the two diagrams, the former as $\sqrt{40}$ and $\sqrt{24}$, and the latter as 6·4 and 4·8, when the diameter of the circles = 8, and I thought it *possible* you *might* refer me to an inscribed square to a circle, the length of the sides being $4\sqrt{2}$, and the breadth $4\sqrt{2}$, as you give them. Now, I doubt whether you can construct a geometrical figure, in which you can shew—isolated and exhibited—any of the rectangles you have given me, *the first and last excepted.* The "*onus probandi*" rests with you to prove this. You will not dare to dispute, that we may circumscribe a circle about the rectangle F B P H, in the enclosed diagram; and I have proved in my last Letter that when the diameter of the circle X = 8, the sides of this rectangle may be, and are, 3 and 3·875, by construction. And I doubt whether it ever entered into your mind to construct such a rectangle, or that such a rectangle could be geometrically constructed. Well, then, the area of this rectangle is 3 × 3·875 =

11·625, while you would make it to be only 3 ($\sqrt{15}$) = 11·618, &c. The difference is sufficient to prove the fallacies into which we may be led by *mis-applied* Mathematics. In your Letter of the 2nd December, with reference to the admissions you had made in your previous Letter, you very *curtly* put the question :—"*But now that you have the admissions what will you do with them?*" Taking this and previous Letters in connection, you cannot fail to perceive that I have made some use of them.

With reference to the enclosed diagram, the following things are self-evident from mere inspection. First: The angle M, in the right-angled triangle K B M, is outside the circle X. Second: The angle H, in the right-angled triangle H F B, is outside the line K M. Third: *m n* = *m* F, for they are radii of the circle Z. Fourth: B T = B F, for they are radii of the circle X Y. Fifth: If the lines K H and B M be produced to meet at a point, say X, and so construct a right-angled triangle K B X, the angle X will be further outside the circle Z than the angle M.

Now, because K M is parallel to O T, and because K B is bisected at O, and B F = B T = $\frac{3}{8}$ (K B) or $\frac{3}{4}$ (O B), by construction: it follows, that B M, the base of the right-angled triangle K B M, is bisected at T, and that K B M and O B T are similar right-angled triangles. Again: B T = $\frac{3}{4}$ (O B), and, B M = $\frac{3}{4}$ (K B), by construction; and because F *m* = $\frac{3}{4}$ (K F), by construction, it follows, that K F *m*, K B M, and O B T, are similar right-angled triangles, and that F *m* is a tangent to the circle X Y: B M a tangent to the circle X: and F K and F B tangents to the circle Z. Again: Because B F = B T = $\frac{3}{4}$ (B O) or $\frac{3}{8}$ (B K), by construction, F K = $\frac{5}{3}$ (B F); therefore, $BF^2 + BO^2 = FK^2$. But, B T = B F; therefore, $BT^2 + BO^2 = FK^2$. But, $BT^2 + BO^2 = OT^2$; and it follows, that O T = F K.

So far, all is pure Geometry, but we must now bring Mathematics to bear, in further considering the properties of this extraordinary geometrical figure.

Well, then, let K B = 8. Then: O B = $\frac{1}{2}$ (K B) = 4: F B = B T = $\frac{3}{8}$ (K B) = 3: K F = $\frac{5}{8}$ (K B) = 5: F *m* = $\frac{3}{4}$ (K F) = 3·75: T P = $\frac{7}{24}$ (B T) = ·875: B M = 2 (B T) = 6. But, V H = B T, and H F = B P, for they are opposite sides of parallelograms;

therefore, $FV = TP = \cdot875$. But, $mF - VF = mV$; therefore, $3\cdot75 - \cdot875 = 2\cdot875 = mV$.

Now, mFB is a right-angled triangle; therefore, $(mF^2 + FB^2) = (3\cdot75^2 + 3^2) = (14\cdot0625 + 9) = 23\cdot0625 = mB^2$; therefore, $\sqrt{23\cdot0625} = mB$. But, BFV is a right-angled triangle; therefore, $(BF^2 + FV^2) = (3^2 + \cdot875^2) = (9 + \cdot765625) = 9\cdot765625 = BV^2$; therefore, $\sqrt{9\cdot765625} = 3\cdot125 = BV$. Now, BVm, a part of the right-angled triangle BFm, is an oblique-angled triangle, and by Euclid: Prop. 12: Book 2: $\{BV^2 + Vm^2 + 2(Vm \times VF)\} = Bm^2$; that is, $\{3\cdot125^2 + 2\cdot875^2 + 2(2\cdot875 \times \cdot875)\}$, or, $\{9\cdot765625 + 8\cdot265625 + 5\cdot03125\} = 23\cdot0625 = Bm^2$: and so far, Euclid would *not* appear to be at fault in the 12th Proposition of his Second Book. But, KFm is a right-angled triangle; therefore, $(KF^2 + Fm^2) = (5^2 + 3\cdot75^2) = (25 + 14\cdot0625) = 39\cdot0625 = Km^2$; therefore, $\sqrt{39\cdot0625} = 6\cdot25 = Km$. But, KBM is also a right-angled triangle, and similar to the triangle KFm; therefore, $KB^2 + BM^2 = (8^2 + 6^2) = (64 + 36) = 100 = KM^2$; therefore, $\sqrt{100} = 10 = KM$. But, $mn = mF$, for they are radii of the circle Z, and this would make $Km + mn = 6\cdot25 + 3\cdot75 = 10$; and so make $Kn = KM$, which is absurd.

Again: BmM is a right-angled triangle, and I have proved that $Bm^2 = 23\cdot0625$, when KB, the diameter of the circle X, $= 8$; therefore, $(BM^2 - Bm^2) = (6^2 - \sqrt{23\cdot0625^2}) = (36 - 23\cdot0625) = 12\cdot9375 = mM^2$; therefore, $\sqrt{12\cdot9375} = 3\cdot59$, &c.; and this would make mM a shorter line than mn, which is absurd.

Again: by analogy or proportion,—according to Euclid—$KF : FB :: Km : mM$; that is, $5 : 3 :: 6\cdot25 : 3\cdot75$. This would make mM and mn equal, which is absurd.

Again: by analogy or proportion,—according to Euclid—$Km : mB :: mB : mM$; that is, $6\cdot25 : \sqrt{23\cdot0625} :: \sqrt{23\cdot0625} : mM$. This analogy worked out by that "*indispensable instrument of science, Arithmetic,*" makes $mM = \sqrt{13\cdot6161}$; that is, $\sqrt{\left(\frac{23\cdot0625^2}{6\cdot25^2}\right)} = \sqrt{\left(\frac{531\cdot87890625}{39\cdot0625}\right)} = \sqrt{(13\cdot6161)} = mM$. But, $\sqrt{13\cdot6161} = 3\cdot69$, exactly. This would make mM less than mn, which is absurd.

On all these *reductio ad absurdum* shewings, Euclid is at fault, in attempting to make his Fifth Book, on proportion, of general and universal application, and therefore true under all circumstances. I might demonstrate other absurdities, but surely this must be unnecessary.

Mr. Wilson observes, in the preface to his Treatise on "Elementary Geometry:" "*Already the Fifth Book (Euclid) has practically gone, and, in consequence, the study of the Sixth Book has become somewhat irrational:*" and, in this, Professor de Morgan agrees with him; for he says, in his review of Mr. Wilson's work:— "*It must be granted that some of his (Euclid's) defects have powerfully aided in introducing a routine of* saying *propositions, without any attention to the meaning. There are great schools and great colleges, in which care is taken that no attention shall be paid to the meaning, by a provision that there shall be no meaning to attend to.*" Is it a matter of wonder, under such circumstances, that there should be found, in "*great schools and great colleges,*" Mathematical teachers who are incompetent to "*meet ignorance with instruction?*"

I had written so far, when this afternoon's post brought me your Letter of yesterday. I had more to say, but I shall pause, and reserve what I should have said for another communication. If you *read* the opening paragraph of this Letter, you will *learn* and *know* the *terms* upon which I shall be prepared to make the apology you "*expect.*" In my Letter of the 19th instant, I replied to every sentence of yours of the 17th, *seriatim*, the last paragraph excepted. It appears to me, that nothing short of admitting that you have met "*ignorance with instruction*" will satisfy you. How can I *conscientiously* admit this, while I remain unconvinced of it? How can I believe it, while I remain under the conviction that neither you nor any other man can prove it? How am I to be convinced, while you meet sound reasoning from indisputable data, with *dogmatical and dictatorial assertion*, without a shadow of proof?

From your reticence with reference to my Letters of the 17th and 19th instant,—passing over that of the 22nd, which you say you have not read—coupled with the declaration in yours of the former date, that you would "*take no notice of any more of my Letters*" until I had done certain things, which I find it difficult to persuade myself you did not believe to be impossible; how could I think otherwise, than that, so far as you were concerned, our correspondence was at an end? You *know*, that although I write *to* you, I do not write *for* you. It may not be in my time, but the day will come, when the "*common sense of mankind*" will pronounce a just verdict between us.

The threat towards the close of your Letter does not disturb me, and to some of the other parts of it, I shall reply at my convenience.

Faithfully yours,
JAMES SMITH.

THE REV. PROFESSOR WHITWORTH.

JAMES SMITH *to* THE REV. PROFESSOR WHITWORTH.

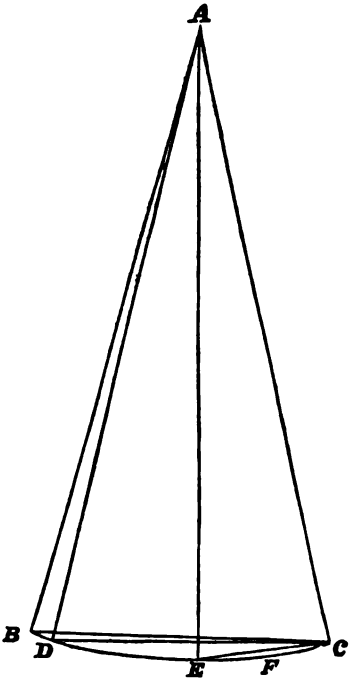
BARKELEY HOUSE, SEAFORTH,
1st January, 1869.
Posted 4th Jan.

SIR,

In your Letter of the 5th December you said: "*Now, if we are either of us to be the wiser for our correspondence, it can only be by following one subject till it is exhausted. The question at present is this: which is the true value of π? As soon as you have either proved $\pi = 3{\cdot}125$, or admitted that $\pi = 3{\cdot}141592\ldots$ then I will answer your question about the constructive geometry, or consider any other subject you please.*" Now, Sir, I have proved over and over again, that $\pi = 3{\cdot}125$, and that it cannot be $3{\cdot}141592\ldots$ "*If you can't see this, I can't help it, but the fact remains notwithstanding.*"

Do you deny that πr^2 = area in every circle? I hardly think you will go so far as this. Well, then, let A B C denote a right-angled triangle, B the right angle, and the side A B the radius of a circle, and be the longer of the sides that contain the right angle. Let the sides that contain the right angle, be represented by the arithmetical symbols 3 and 4; then, the hypothenuse will be 5, and the sum of the squares of the three sides of the triangle exactly equal to $3\frac{1}{8}$ times the area of a square on the middle side A B, which is the radius of the circle. I am almost "*ashamed*" to reiterate this plain and incontrovertible statement. Well, then, $\left\{\left(50-\frac{50}{5}\right)-\frac{1}{8}\left(50-\frac{50}{5}\right)\right\}=\left\{(50-10)-\frac{1}{8}(40)\right\}=(40-8)=32=2(\text{A B}^2)$ = area of an inscribed square to a circle of radius = 4. and $50 \div \frac{3\frac{1}{8}}{4}$, or $50 \times 1{\cdot}28 = 64$ = area of a circumscribing square to the circle: and I have proved over and over again, that $\frac{{\cdot}78125}{1}$ and $\frac{1}{1{\cdot}28}$ are equivalent ratios. In the Letter referred to you say: "*But my patient examination of all your proofs deserves some consideration.*" I have read your Letters with great care, and I can find no evidence that you have ever examined the foregoing proof; and if it stood alone, it would be sufficient to convince any "*reasoning geometrical investigator,*" that 3·125 is the true arithmetical value of π. I have given you this proof of the value of π in two previous Letters, Nov. 23, and Dec. 2, simply giving the converse of this operation; that is to say, making the calculations from the area of a square to the area of its circumscribing circle, and you have never attempted to controvert these proofs, and have not even noticed them in any way. If you can't find the areas of an inscribed and circumscribed square to a circle, by the assigned value of π, (the area of the circle being the given quantity,) or any other value of π, (3·125 excepted,) and you know that you cannot, "*you ought not to be ashamed to say plainly,*" that 3·125 must be the true arithmetical value of π, which makes 8 circumferences = 25 diameters in every circle.

In your Letter of the 30th November (at the time you wrote this Letter you were not convinced by any of the proofs I had then given you that $\pi = \frac{25}{8}$), you made the following request:—"*Will you point me to any other arguments which you can bring forth to prove $\pi = \frac{25}{8}$; If you can give me one single proof (in which I can detect no flaw) that your value of π is correct, I will then scrutinise once more the proofs by which the orthodox value of π is established.*" The "*onus probandi*" rests with you to point out a "*flaw,*" in the foregoing proof: and until you have done so, or admit your inability to do it, I deny that you have any *moral* right to dictatorially demand of me, to controvert your *absurd* "strictures" on the many proofs I have given you.



In your last letter you again reiterate the "*absurd charge*" that I make a xxv.-gon and the circumference of its circumscribing circle equal; and refer me to my Letter of the 23rd November. In this geometrical figure let A B C denote one of 24 isosceles triangles inscribed in a circle: let A D C denote one of 25 isosceles triangles, and A E C one of 50 isosceles triangles, inscribed in a circle. Now, the chord B C, is to the arc B E C, as the chord D C, to the arc D E C; and the chord D C, to the arc D E C, as the chord E C, to the arc E F C: and the ratio of chord to arc is an invariable ratio. Hence, as I have distinctly proved, that into whatever number of arcs we may divide the circumference of a circle, if from one of these arcs we deduct one twenty-fifth part, and multiply the remainder by the sum of the arcs, the product is a constant quantity, and equal to the perimeter of a regular hexagon inscribed in the circle. I have gone into this at some length in my Letter of the 11th December. When, or where —*the proverb of Solomon, with which the Letter concludes excepted* —have you ever taken the slightest notice of that communication? Can this be called fair controversy? Were you ever taught, either at school or college, or did it ever occur to you, that the circular measure of an angle of 60° $= \frac{1}{6}(2\pi)$; the circular measure of an angle of 30° = $\frac{1}{12}$ (2 π); the circular measure of an angle of 15° = $\frac{1}{24}$ (2 π), whatever be the value of π? I doubt it! But it is so, and it follows of necessity, that the circular measure of an angle of 14° 24′ = {$\frac{1}{25}$ (2 π),} whatever be the value of π! For example: by hypothesis, let B A C denote an angle of 30°; and π = 3·1416. Then: $\frac{30^\circ \times \pi}{180} = \frac{30 \times 3{\cdot}1416}{180} = \frac{3{\cdot}1416}{6} = {\cdot}5236$ = the circular measure of the angle B A C, and is equal to the arc B E C, when A B and A C = 1; and 12 (·5236) = 6·2832 = 2 π. Now, every tyro in Geometry and Trigonometry *knows*, that the area of a circle of radius = 1, and the circumference of a circle of diameter unity, are represented by the same arithmetical symbols, whatever be the value of π. But, $\left({\cdot}5236 - \frac{{\cdot}5236}{25}\right)$ = (·5236 — ·020944) = ·502656, and is greater than the radius of a circle of diameter unity; but you will observe, that ·502656 is equal to one-hundredth part of the area of a circle of radius 4, on the hypothesis that π = 3·1416.

Now, let B A C denote an angle of 30°; and, by hypothesis, let π = $\frac{25}{8}$ = 3·125. Then: $\frac{30^\circ \times \pi}{180} = \frac{30 \times 3{\cdot}125}{180} = \frac{3{\cdot}125}{6}$ = ·5208333, with 3 to infinity, = $\frac{25}{24}$ (·5); or, in other words, is equal to twenty-five twenty-fourth parts of the radius of a circle of diameter unity; and is the circular measure of the angle B A C, and equal to the arc B E C: and, 12 (·5208333 . . .) = 6·24999996 . . . If you tell me that this does not stand for 2 π, on the hypothesis that π = $\frac{25}{8}$ = 3·125, I have simply to call your attention to the fact, that if you work out the calculations by your value of π = 3·141592, you will find yourself beset with the same difficulty. Surely I need not remind you that this simply arises from π not being divisible by 6, or the multiples of 6, without a remainder, whether we adopt π = 3·125, or π = 3·141592.

On the former hypothesis, we cannot connect the circular measure of the angle with any thing: the latter hypothesis makes $\frac{4}{25}$ (circumference) equal to an arc subtending an angle at the centre of a circle, equal to radius: and you will surely not dispute that circular measure is of "*great importance in the theory of Mathematics.*"

Again: Let D A C denote an angle of 14° 24′: and, by hypothesis, let $\pi = 3{\cdot}125$. Then: $\frac{14^\circ\, 24' \times \pi}{180} = \frac{864' \times \pi}{180 \times 60} = \frac{2700}{10800} =$ ·25 = the circular measure of the angle D A C, and is equal to the arc D E C, when A D and A C = 1. $(\cdot 25 - \frac{\cdot 25}{25}) = (\cdot 25 - \cdot 01) =$ ·24 = the chord D C: and, the chord D C : arc D E C : : the chord B C : the arc B E C; that is, ·24 : ·25 : : ·5 : ·5208333... ; or, ·24 : ·25 : : 3 : 3·125 ; or, 3 : 3·125 : : ·5 : ·5208333... Now, according to your reasoning, $\frac{B\,C}{2} = \frac{\cdot 5}{2} =$ ·25 should be the natural sine of the angle B A C. But, the angle B A C is an angle of 15°, and the natural sine of this angle is given in Hutton's Tables as ·2588190. Now, conceive a straight line, M N, of the same length as A B, to revolve from the "*initial position,*" A B, in the direction of A C, until it coincides with A C. Conceive further, that the line M N is arrested at different stages in the course of its revolution, at various points in the arc B E C, and straight lines drawn from the point C to these points. It is self-evident that these divergent lines from the point C, will be the chords of various arcs. Now, is it not self-evident, that the chords vary as the arcs, and that the ratio of chord to arc is an invariable ratio? Well, then, this effect—and there can be no effect without a cause—may be said to arise from a "*quantity of turning,*" applied to the extremity of a straight line revolving round a curved line: and explains other things to which I have directed your attention in previous parts of this communication. These facts ought to be sufficient to satisfy you that the trigonometrical functions of angles are not lengths, but ratios of one length to another, and ought to convince you of the absurdity of your arguments and conclusions.

Again: Let the angle E A C = half the angle D A C, and by hypothesis, let $\pi = 3{\cdot}125$. Then: E A C denotes an angle of 7° 12′; and $\frac{7^\circ\, 12' \times \pi}{180} = \frac{432' \times \pi}{180 \times 60} = \frac{1350}{10800} =$ ·125, and ·125 is the circular measure of the angle E A C, and the length of the arc E F C, when A E and A C = 1. $\left(\cdot 125 - \frac{\cdot 125}{25}\right) = (\cdot 125 - \cdot 005) = \cdot 12 =$ the chord E C: and the chord E C : the arc E F C : : the chord D C : the arc D E C, that is, ·12 : ·125 : : ·24 : ·25 : or, the chord E C : the arc E F C : : the chord B C : the arc B E C; that is, ·12 : ·125 : : ·5 : ·52083333, with 3 to infinity. Thus: 50 (·12) = 6 = perimeter of a regular inscribed hexagon to a circle of radius 1. $3 : \pi : : 6 : 2\pi$; that is, 3 : 3·125 : : 6 : 6·25; and 6·25 is the circumference of a circle of radius 1. 50 (half the arc E F C) = $50\left(\frac{\cdot125}{2}\right)$ = 50 (·0625) = 3·125 = area of a circle of radius 1. But, circumference × semi-radius = area in every circle; therefore, (6·25 × ½) = 6·25 × ·5 = 3·125 = area of a circle of radius 1. These facts will be obvious enough to any "*reasoning geometrical investigator*," and ought to be sufficient to convince you of the absurdity of your arguments and conclusions. Can you produce these results with your value of π = 3·141592...? I know that you cannot! Have I not then a right to expect you *at once* to withdraw the "*absurd charge*" you have brought against me, that I make a certain chord and arc equal? It appears to me that I have a far greater show of reason for expecting an apology from you, than you have for expecting an apology from me, for anything I have written.

I admit that you are a "*recognised Mathematician*," I *assume* you "*to be a gentleman*" and "*a man of honour*:" and I *know* you to be a minister of the gospel. I *suppose* I know something of mathematics, and I claim to be a gentleman and a man of honour, and if not a clergyman, I am so far a lay theologian, that I have, in my time, edited a volume of sermons. I may observe, that there is one point upon which I do not require instruction: I have learned, and know, and bear in mind, that if I violate or tamper with conscience, there is ONE, "*who doeth as he will in the armies of heaven, and amongst the inhabitants of the earth*," and who will not permit me to escape with impunity.

I had written so far when it occurred to me to refer to the copy of my Letter of the 23rd November, and I was certainly astonished to find the blunder on page 10. $\frac{360}{25}$ is obviously the value of an *arc* subtending the angle B A C at the apex of the triangle, and not the value of the *chord* B C subtending that angle. I frankly admit this *lapsus*, but shall defer any further reference to it for another communication.

Faithfully yours,

JAMES SMITH.

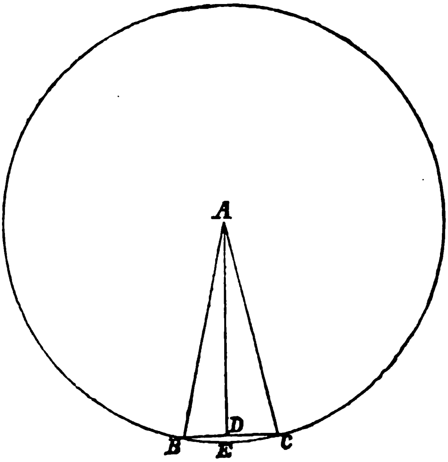
THE REV. PROFESSOR WHITWORTH.

JAMES SMITH *to* THE REV. PROFESSOR WHITWORTH.

BARKELEY HOUSE, SEAFORTH,
4th January, 1869. *Posted* 11*th.*

SIR,

Since the trigonometrical functions of angles are not lengths, but ratios of one length to another, it follows, that the sine of an angle at the centre of a circle contained by two radii, has to do *directly* with the arc that subtends it, and only *indirectly* with its subtending chord: and I have proved that whatever number of regular polygons may be inscribed in a circle, the ratio between the sides of the polygons and their subtending arcs is an invariable ratio, and is as ·24 : ·25, or, as 3 : 3·125. Now, it is self-evident that if 6, 12, and 24-sided polygons be inscribed in a circle, the length of the chord to its subtending arc is not *arithmetically* in the same ratio in the 12-sided as in the 6-sided polygon: and that the ratio between the length of the chord and its subtending arc in the 24-sided polygon differs from both. In other words, the length of the chord in a 12-sided polygon is greater in proportion to its subtending arc, than the length of the chord in a 6-sided polygon to its subtending arc; and the length of the chord in a 24-sided polygon is greater in proportion to its subtending arc than either. Hence, the inapplicability (directly) of the 47th Proposition of Euclid's first book, to the measurement of a curved line.



In this geometrical figure, let A B C denote one of 25 equal isosceles triangles inscribed in the circle, with the angle at the apex and the chord B C bisected by the line A D.

Now, the circular measure of an angle of 1 degree is $\frac{\pi}{180}$; and, $\frac{180}{\pi}$ is the number of degrees contained in the angle which is subtended by an arc equal to radius, whatever be the value of π. But, 6 times $\frac{4}{25}$ (circumference) is the perimeter of a regular inscribed hexagon to a circle, whatever be the circumference. For example: Let the circumference of the circle $= 2\,\pi$; and, by hypothesis, let $\pi = 3{\cdot}141592$. Then: $2\,(3{\cdot}141592) = 6{\cdot}283184 =$ circumference, when A B and A C $= 1$; and $\frac{4}{25}\,(6{\cdot}283184) = \frac{4 \times 6{\cdot}283184}{25} = 1{\cdot}00529984$. Now, $2\,\pi$ is the circumference of a circle of radius 1, and 6 (radius) = the perimeter of a regular inscribed hexagon to every circle. But, $6\,(1{\cdot}00529984,) = 6{\cdot}03179904$, and is greater than 6, the known and indisputable value of the perimeter of a regular inscribed hexagon, to a circle of radius 1; and it follows, that $2\,(3{\cdot}141592) = 6{\cdot}283184$, is greater than the true circumference of a circle of radius 1. But further, 6 times $\frac{4}{25}$ (circumference) $=$ $\frac{24}{25}$ (circumference); that is, $6\left(\frac{4 \times 6{\cdot}283124}{25}\right) = \frac{24}{25}\,(6{\cdot}283124)$, or, $(6 \times 1{\cdot}00529984) = \left(\frac{24 \times 6{\cdot}283124}{25}\right)$, and this equation or identity $= 6{\cdot}03179904$; and this is the true arithmetical value of the perimeter of a regular inscribed hexagon to a circle of circumference $=$ $6{\cdot}283124$. Hence: $3{\cdot}125 : 3 :: 6{\cdot}283124 : 6{\cdot}03179904$, and proves that $3{\cdot}125$ must be the true arithmetical value of π.

Now, $\frac{24}{25}\,(100) = \left(\frac{24 \times 100}{25}\right) = 96 =$ perimeter of a regular inscribed hexagon to a circle of circumference $= 100$. $\frac{24}{25}\,(360) = \left(\frac{24 \times 360}{25}\right) = 345{\cdot}6 =$ perimeter of a regular inscribed hexagon to a circle of circumference $= 360$. And $\frac{24}{25}\,(6{\cdot}25) = \frac{24 \times 6{\cdot}25}{25} =$ $6 =$ perimeter of a regular inscribed hexagon to a circle of circumference $= 6{\cdot}25$, or radius $= 1$. But, $\frac{96 \times 6}{10} = \frac{345{\cdot}6}{6}$, that is, $\frac{576}{10} = \frac{345{\cdot}6}{6}$, and this equation or identity $= \frac{180}{\pi} = \left(\frac{180}{\frac{1}{2}\,(6{\cdot}25)}\right)$ $= \frac{180}{3{\cdot}125} = 57{\cdot}6$, and this is the number of degrees contained in the angle which is subtended by an arc equal to radius, when the circumference of the circle $= 360$.

Now, referring to the diagram, let the circumference of the circle be represented by the number 360. Then $\frac{360}{25} = 14{\cdot}4$, gives the angle

B A C; that is to say, gives the angle at the apex of the triangle A B C, which is an angle of 14° 24′ at the centre of the circle: and it follows, that the arc B E C subtending the angle B A C is an arc of 14° 24′. But, the angle B A C is bisected by the line A D, and it follows, that the angles D A B and D A C, are angles of 7° 12′; and half the arc B E C $= \frac{14 \cdot 4}{2} = 7 \cdot 2$; and A D B and A D C are similar and equal right-angled triangles. Now, $\left(7 \cdot 2 - \frac{7 \cdot 2}{25}\right) = (7 \cdot 2 - \cdot 288)$ $= 6 \cdot 912 = \frac{24}{25}(7 \cdot 2) =$ D B and D C $= \frac{1}{2}$(B C): or, I should rather say, 6·912 to 7·2 is the ratio between half the chord B C, and half the arc B E C; and is in the ratio of 3 to 3·125; that is, 3 : 3·125 :: 6·912 : 7·2.

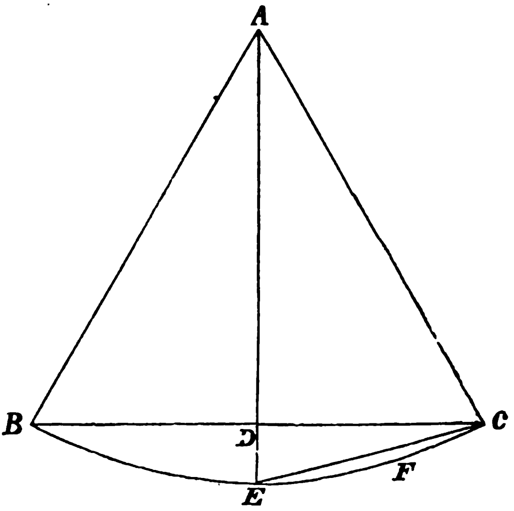
By hypothesis, let half the chord B C be greater than 6·912, but less than half the arc B E C, say 7·15. Then: $(\text{A B}^2 - \text{B D}^2) =$ $(57 \cdot 6^2 - 7 \cdot 15^2) = (3317 \cdot 76 - 51 \cdot 125) = 3266 \cdot 6375 = \text{A D}^2$; therefore, $\sqrt{3266 \cdot 6375} = 57 \cdot 1545 \ldots =$ A D. But, $\frac{\text{B D}}{\text{A B}} = \frac{7 \cdot 15}{57 \cdot 6} = \cdot 12413\ldots$ = the Sin. of the angle D A B: and $\frac{\text{A D}}{\text{A B}} = \frac{57 \cdot 1545}{57 \cdot 6} = \cdot 9922656$ = the Cos. of the angle D A B; the former *less*, and the latter *greater*, than the natural sine and cosine of an angle of 7° 12′ as given in Tables.

Again: Let the circumference of the circle be represented by the number 360; and, by hypothesis, let $\pi = 3 \cdot 141592$. Now, the angles D A B and D A C are indisputably angles of 7° 12′, and $\frac{\pi}{180}$ is the circular measure of an angle of 1 degree $= \frac{3 \cdot 141592}{180} =$ $\cdot 01745328$, on the hypothesis that $\pi = 3 \cdot 141592$: therefore, $\cdot 01745328 \times 7° \, 12' = \cdot 01745328 \times 7 \cdot 2 = \cdot 125667616$, and is greatly in excess of the trigonometrical sine of the angles D A B and D A C.

I shall now proceed to demonstrate that the trigonometrical sines and circular measure of the angles D A B and D A C are equal; that is to say, that the trigonometrical sines of the angles D A B and D A C are exactly equal to the circular measure of half the arc B E C. Now, the circular measure of the angles D A B and B A C is $\frac{7° \, 12' \times \pi}{180}$

$= \frac{432' \times 3{\cdot}125}{180 \times 60} = \frac{1350}{10800} = {\cdot}125$. But, $\frac{\pi}{180} = \frac{3{\cdot}125}{180} = {\cdot}071361111$ with 1 to infinity, is the circular measure of an angle of 1 degree; therefore, $\cdot071361111 \ldots \times 7^\circ\ 12' = {\cdot}071361111 \times 7{\cdot}2 = {\cdot}1249999992 \ldots$, which represents to us ·125, and is the trigonometrical sine and circular measure of the angles D A B and B A C. But, $\frac{7{\cdot}2}{57{\cdot}6} = {\cdot}125$, and proves that if equal isosceles triangles be inscribed in a circle, the angles at the centre of the circle are *directly* connected with the arcs that subtend them, and only *indirectly* with their subtending chords; or, in other words, the natural sine of half the angle B A C, is equal to the circular measure of half the arc B E C that subtends it; and half the chord B C is to half the arc that subtends it, in the ratio of 6·912 to 7·2, or, in the same ratio as the perimeter of a regular hexagon, to the circumference of its circumscribing circle.

In this geometrical figure, let A B C be an equilateral and equiangular triangle, with the angle B A C at the apex and its opposite side B C, bisected by the line A D. Let C E be a straight line drawn from the angle C to meet A D produced at the point E, bisecting the arc B E C contained by A B and B C, the legs of the isosceles triangle A B C.



Now, it is self-evident that the arc B E C is one-sixth part of the circumference, and the arc E F C one-twelfth part of the circumference, to a circle of which A B and A C are radii. Conceive a "*quantity of turning*" applied to the line C E, in the direction of D, until it coincides with C B. Is it not self-evident that the extremity E of the line C E will rest on the line B C at a point intermediate between B and D? Is it not therefore self-evident that the line C E is longer than half the line B C? But, the arc E F C is half the arc B E C, and it follows, that the chord E C is longer in proportion to *its* subtending arc E F C than the chord B C to *its* subtending arc B E C.

Let A B and A C = 1. Then: $\frac{1}{6}(2\pi) = \frac{6{\cdot}25}{6}$ = 1·0416666 with 6 to infinity. $\frac{24}{25}(1{\cdot}0416666) = \frac{24 \times 1{\cdot}0416666...}{25}$ = ·999999 &c., and it is self-evident that by extending the decimals we should get 9 to infinity. Does it not follow that ·999999 &c., must represent to us 1, as certainly as that the infinite series "$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ &c.," must represent to us 2? And does it not follow, that 1 to 1·0416666 with 6 to infinity is the ratio between the chord B C and its subtending arc B E C, since 6 times 1 = 6, is the known and indisputable value of the perimeter of a regular inscribed hexagon to a circle of radius 1? Again: $\frac{1}{12}(2\pi) = \frac{6{\cdot}25}{12}$ = ·52083333 with 3 to infinity, is the value of the arc E F C, and is equal to $\frac{25}{24}$th parts of the radius of a circle of diameter unity. $\left\{{\cdot}52083333 - \frac{{\cdot}52083333}{25}\right\} = \left\{{\cdot}52083333 - {\cdot}02083333\right\} = {\cdot}5$ = radius of a circle of diameter unity; and 6 (·5) = 3, is the perimeter of a regular inscribed hexagon to a circle of diameter unity. But, 12 (·5) = 6, is the perimeter of a regular inscribed hexagon to a circle of radius 1, and it follows, that ·5 to ·52083333 with 3 to infinity, is the ratio between the chord E C and its subtending arc E F C. We might bisect the angle D A C and its subtending chord and arc, and join C F. Then: the arc C F would equal $\frac{1}{24}(2\pi)$ = ·260416666, with 6 to infinity, and make the ratio of the chord C F to its subtending arc as ·25 to ·260416666, with 6 to infinity, or as 3 to π: and so on we might proceed, *ad infinitum.* From these facts we obtain the following remarkable result, in which every other but the true arithmetical value of π fails: $\frac{25}{24}(3\pi) = \frac{25}{24}(9{\cdot}375) = \frac{25 \times 9{\cdot}375}{24} = \frac{234{\cdot}375}{24} = \pi^2 = 3{\cdot}125^2 = 9{\cdot}765625$.

Now, when A B and A C = 1, D C = $\frac{1}{2}$ = ·5, and A D = (A C^2

— D C^2) = (1^2 — ·5^2) = (1 — ·25) = ·75 = A D^2; therefore, A D = $\sqrt{\cdot75}$ = ·8660254 : and it follows, that D E the versed sine of the angle D A C = 1 — ·8660254 = ·1339746. Hence : (D C^2 + D E^2) = (·5^2 + ·1339746^2) = (·25 + ·017949193445 16) = ·267949193445 16 = (C F^2) ; therefore, $\sqrt{\cdot26794919344516}$ = ·517648 = C E. This makes the chord C E to its subtending arc, not in the ratio of ·5 to $\frac{25}{24}$ (·5), but in the ratio of ·517638 to $\frac{1}{12}$ (2π), whatever be the value of π. Hence, the inapplicability of the 47th Proposition of Euclid's first book, to measure *(directly)* a curved line : but I have proved in many ways that *indirectly* this proposition is of the utmost value and importance, in aiding us to find the value of π, and the true ratio of diameter to circumference in a circle. Is not $\frac{\cdot517638}{2}$ = ·2588190, the natural sine of an angle of 15°, as given in Tables ?

Now, 12½ times ½ = 12·5 × ·5 = 6·25 = 2π ; and 12½ times (¼)^2 = 12·5 × ·25^2 = 12·5 × ·0625 = ·78125 = $\frac{6\cdot25}{8}$, and represents the area of a circle of diameter unity = $\frac{\pi}{4}$, and is exactly equal to 3⅛ (·5^2). Hence : 12½ times $\frac{\pi}{50}$ = 3⅛ times the area of a square on the radius of a circle of diameter unity ; and it follows, that 4π ($s\ r^2$) = area in every circle. Proof : Let the area of a square on the semi-radius of a circle be represented by any arithmetical quantity, say 60. Then : 4π (60) = 12·5 × 60 = 750 = area of the circle, $\left\{\left(750-\frac{750}{5}\right) - \frac{1}{5}\left(750-\frac{750}{5}\right)\right\}$ = (600 — 120) = 480 = area of an inscribed square to the circle. 2 (480) = 960 = area of a circumscribing square to the circle ; therefore, the diameter of the circle = $\sqrt{960}$. $\frac{\sqrt{960}}{4} = \sqrt{960 \div 4^2} = \sqrt{\frac{960}{16}} = \sqrt{60}$ = semi-radius of the circle : and, $\sqrt{60}^2$ = 60 = the given area of the square on the semi-radius. Now, Sir, can you produce this result by the value of π arrived at by "*recognised Mathematicians?*" Certainly not, *and you know it!* Well, then, "*you ought not to be ashamed to admit*" that $\frac{25}{8}$ = 3·125, is the true value of π. As a "*recognised Mathematician*" you can readily convince yourself of this fact, and as a "*gentleman and a man of honour,*" it becomes your duty to admit it, if truth, and the maintenance and advancement of scientific truth, be your sole object in your controversy with me.

I have proved that when A B and A C in the figure on page 156 = 1, E C the chord of the arc E F C = ·517638 approximately. Now, 12(·517638...) = 6·211656... and if you compute the value of the sides of a 24-sided polygon by the 1st and 47th of Euclid, the sum of the sides will increase this quantity: and if you compute the value of the sides of a 48-sided polygon, the sum of the sides will be still greater; and proceeding in this way you will at length approximate to the value of 2π as assigned by "*recognised Mathematicians.*" Well, then, the perimeters of all polygons of more than 6 sides are incommensurable, and it is self-evident that π cannot be arithmetically either finite or determinate, if the "*recognised Mathematician's*" principle of finding the arithmetical value of π be a sound one. But, you say you teach that "π *is finite and determinate,*" and attempt to make it so, by putting a *fanciful* interpretation upon the term "*infinite series.*" This goes to the root of the *fallacious assumption*, by which "*recognised Mathematicians*" are led to a false value of π. I presume you know that the perimeter of a regular hexagon is to the circumference of its circumscribing circle, as the area of an inscribed regular dodecagon to the area of the circumscribing circle: and with your "*logical mind*" you *ought* to be able to see, that it follows of necessity, that the perimeter of a 12-sided regular polygon is to the circumference of its circumscribing circle, as the area of a 24-sided regular polygon to the area of the circle. To controvert this fact, you must prove, that as we increase the number of sides of an inscribed polygon to a circle, we do not increase the areas of the polygons in the same ratio. The truth is, "*recognised Mathematicians,*" in *assuming* that they can ascertain the approximate value of π by means of polygons, whether inscribed or circumscribed to a circle, utterly ignore the fact, that a line in the form of a circle, encloses a larger area than it can be made to enclose in any other form whatever.

I must now revert to the diagram on page 153. Let BAC denote an angle of 14° 24', and, by hypothesis, let $\pi = 3{\cdot}1416$. Then: $\frac{14^\circ\,24' \times \pi}{180} = \frac{864' \times 3{\cdot}1416}{180 \times 60} = \frac{2714{\cdot}3424}{10800} = {\cdot}251328$, and is equal to $\frac{8(\pi)}{100}$; or in other words, is equal to the circumference of a circle of diameter $= \frac{8}{100} = {\cdot}08$, on the hypothesis that $\pi = 3{\cdot}1416$. The circular measure of an angle of 90°, that is, the circular measure of a right angle, is $\frac{90 \times \pi}{180} = \frac{90 \times 3{\cdot}1416}{180} = \frac{3{\cdot}1416}{2} = 1{\cdot}5708 = \frac{\pi}{2}$, and is equal to the quadrant of a circle of radius 1, or the semi-circumference of a circle of diameter unity, on the hypothesis that $\pi = 3{\cdot}1416$. Now, $\frac{1{\cdot}5708}{{\cdot}251328} = 6{\cdot}25$, and is not equal to the orthodox value of 2π, but equal to the circumference of a circle of radius 1, on the *theory* that 8 circumferences of a circle are exactly equal to 25 diameters. But, 6·25 is a constant quantity, by whatever "*finite and determinate*" hypothetical value of π you may work out the computation; and of this fact you may readily convince yourself. What, then, in the name of common sense, can the value of π be, but $\frac{25}{8} = 3{\cdot}125$? In how many ways have I proved 3·125 to be the value of π by constructive geometry? When, or where, have you ever attempted to grapple with any of my proofs? Throughout our controversy you have assumed, that it was quite sufficient for you as a "*recognised Mathematician*," to meet demonstration with the assertion :—"*Your arguments are unsound, and your conclusions fallacious*," without a shadow of proof: but the day will come when the common sense of mankind will decline to accept your "*ipse dixit*" for established truth; and repudiate your assumption that on a mathematical question, a "*recognised Mathematictan*" may constitutute himself both "*judge and jury*" in his own cause.

With reference to the following arguments, I must ask you to bear in mind the sixth definition of your "*College chum*," Mr. J. M. Wilson, in his recently published treatise on "*Elementary Geometry.*" Now, conceive a "*quantity of turning*" to be applied to the lines A B and A C—the legs of the isosceles triangle A B C, in the figure on page 153—in opposite directions, until the arc B E C, subtending the angle B A C, should be drawn into a straight line. Then, A B E C would become an isosceles triangle, and exactly equal to one of 24 equal isosceles triangles inscribed in the circle;

if, previous to this *conceivable* operation, A B C denote one of 25 equal isosceles triangles inscribed in the circle. Under these *conceivable* circumstances, the angle at the apex of the isosceles triangle A B E C would become subtended by an arc of 15°: and, dispensing with the symbol that represents a degree, $(15 - \frac{15}{25}) = (15 - {\cdot}6) = 14{\cdot}4$; and this would be the value of the chord subtending the angle at the apex of the triangle A B E C. Hence: $\frac{14{\cdot}4}{2} = 7{\cdot}2 =$ half the chord subtending the angle at the centre of the circle: and radius is to half the chord in the ratio of 57·6 to 7·2; or, in other words, in the ratio of 8 to 1. To controvert this, you must prove that an arc equal to radius is not 57·6, when the circumference of a circle = 360. How will you set about it? You must not take for granted that the series $\frac{16}{\sqrt{3}}\left(\frac{1}{1 \cdot 3} + \frac{2}{5 \cdot 7} \cdot \left(\frac{1}{9}\right) + \frac{3}{9 \cdot 11} \cdot \left(\frac{1}{9}\right)^2 + \right.$ &c.,$\left.\right) = \pi$. This is the question in dispute: and it is for you to furnish the proof. But, even taking for granted that this series represents the value of π, you cannot find, by means of it, the value of the arc which subtends an angle at the centre of the circle, whether you assume the circumference of the circle to be represented by 360, or by π, or by $2\,\pi$.

Now, $\frac{360}{24} = 15$: $\frac{24}{25}(15) = 14{\cdot}4$, and $25\,(14{\cdot}4) = 345{\cdot}6 =$ the perimeter of a regular inscribed hexagon to a circle of circumference $= 360$. $\frac{\pi}{3}\,(345{\cdot}6) = \frac{3{\cdot}125 \times 345{\cdot}6}{3} = \frac{1080}{3} = 360 =$ circumference: and it follows, that $4\,(14{\cdot}4) = \frac{24}{25}\left(\frac{360}{6}\right)$; and this equation or identity $= 57{\cdot}6$, and is the number of degrees contained in an angle which is subtended by an arc equal to radius, when the circumference of the circle = 360.

Again: $\frac{3{\cdot}125}{25} = {\cdot}125$: $\frac{24}{25}({\cdot}125) = {\cdot}12$, and $25\,({\cdot}12) = 3 =$ the perimeter of a regular inscribed hexagon to a circle of diameter unity: and it follows, that $\frac{25}{24}(3) = \frac{25 \times 3}{24} = \frac{75}{24} = 3{\cdot}125 = \pi$. But, $\frac{360}{\pi} = \frac{360}{3{\cdot}125} = 115{\cdot}2 =$ the diameter of a circle of circumference = 360. Therefore, $3\left(\frac{\text{diameter}}{6}\right) = 3\left(\frac{115{\cdot}2}{6}\right) = 3 \times 19{\cdot}2 = 57{\cdot}6$; and again proves that $57{\cdot}6 = 57° 36'$ is the number of degrees contained in the angle which is subtended by an arc equal to the radius, when the circumference of the circle = 360.

Well, then, do you imagine that you can resolve demonstrations like these into absurdity, by boldly—I might even use a stronger term—making such assertions as:—"*Your arguments are unsound, and your conclusions fallacious*," without a shadow of proof?

I shall send you with this communication a copy of my Letter to His Grace the Duke of Buccleuch, ex-President of "*The British Association for the Advancement of Science*," and if you will take the trouble to read from about page 42 to page 53, you will find I have proved, that in small angles, the *trigonometrical* sine may actually be greater than the *circular measure* of the angle: and if you have not resolved to take your stand in the ranks of that numerous class who "*despise wisdom and instruction*," you will not be slow to avail yourself of the opportunity of acquiring this crumb of geometrical and mathematical knowledge.

This brings me to the *lapsus* in my Letter of the 23rd November. On referring to your two Letters in reply to that communication, dated 28th November, I find that you disputed the proof I gave by Logarithms on page 12, that in a right-angled triangle of which the acute angle is an angle of 7° 12', the ratio of hypothenuse to base or shortest side, is as 8 to 1 (and this fact is incontrovertible), and entirely overlooked, *or at any rate passed by unnoticed*, the real blunder, which is on page 10. Whether you had a design in this, or whether it was purely accidental, you know best. Be this as it may, I was ignorant of the *lapsus*, until you drew my especial attention in your Letter of the 28th December, to mine of the 23rd November, which led me to read my copy of the latter, and make the discovery. Had you done this at an earlier period, I should have detected the *lapsus*; at once admitted it; and then, have endeavoured to make my meaning plain and intelligible; as it is, the use you have now made of it, falls upon my mind more like the last effort of "*a drowning man catching at a straw*," than anything else.

Well, then, I admit the *lapsus* on page 10 of my Letter of the 23rd November, but not in the sense you attempt to fasten upon me, as I shall shew you. In an isoceles triangle inscribed in a circle, half the chord subtending the angle at the centre of the circle is the *geometrical* sine of half the angle at the centre; but, (with certain exceptions) is not the *trigonometrical* sine of half the angle at the centre of the circle. You cannot fail to have observed, that in the previous portion of our correspondence I have applied the term "*natural sine*" in the same sense as it is applied by "*recognised Mathematicians*." Throwing this fact out of view—and you have never thought of it—I may admit that my *lapsus* is capable of the construction you have put upon it: but, I find it difficult to persuade myself that you could be convinced in your own mind, that your construction conveyed the idea of my real meaning. In my Letter of the 11th December, I told you distinctly, that in every circle $\frac{25}{8}$ (diameter) = circumference: and $\frac{24}{8}$ (diameter) = the perimeter of a regular inscribed hexagon. When have you ever noticed this? How could you look upon me as a "*gentleman and a man of honour*," if you conscientiously believed me to hold the absurd notion that under any circumstances a chord could be equal to its subtending arc, and yet, that I refused to admit it? Under such circumstances would you have been at the trouble of writing me so many Letters subsequent to those of the 28th November? It may be that you were resolved to write me down by "*power of assertion, not force of reasoning*:" or what is more probable, you thought you would make me lose my temper, and in this way get rid of the difficulty of "*bolstering up*" a bad cause.

It appears to me, that the best way of correcting the blunder I have fallen into in my Letter of the 23rd November, is to substitute what I should have said in lieu of the paragraphs on pages 10 and 11, so as to make my meaning unmistakeably intelligible to any "*reasoning geometrical investigator*." *

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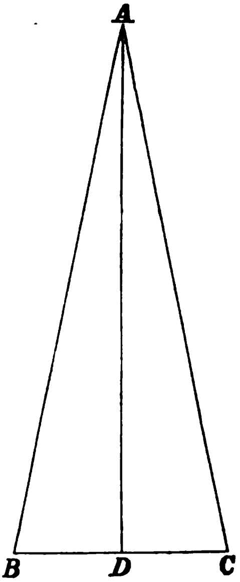
* The reader should take my admission of the *lapsus*, and the explanation I have given of it in the foot-note on page 23, in connection with the three following paragraphs. He will find the statements very different, and yet, quite consistent with each other.



Let A B C denote an isosceles triangle inscribed in a circle, with the angle B A C and its opposite side B C bisected by the line A D. By hypothesis, let the circumference of a circle of which A B and A C are radii = 360.

Now, $\frac{24}{25}$ (360) = 6 (57·6), and this equation or identity, is equal to the perimeter of a regular inscribed hexagon to a circle of circumference = 360 = 345·6. But, perimeter of a regular inscribed hexagon to a circle of diameter unity = 6 ($\frac{1}{2}$) = 6 × ·5 = 3 : and, by analogy or proportion, 345·6 : 360 :: 3 : 3·125. But, $\frac{360}{25}$ = 14·4 = 14° 24′ = the angle B A C, when the isosceles triangle A B C denotes one of 25 equal isosceles triangles inscribed in a circle; therefore, $\frac{14^\circ\ 24'}{2}$ = 7° 12′ is the value of the angles D A B and D A C, when the value of these angles is expressed in degrees: for, the angle B A C and its opposite side B C are bisected by the line A D, and it follows, that A D B and A D C are right-angled triangles.

Well, then, by hypothesis, let A B C denote one of 25 equal isosceles triangles inscribed in a circle of circumference = 360. Then: $\frac{360}{25}$ = 14·4 = an arc subtending the angle B A C, and therefore subtending the chord B C. $\frac{24}{25}$(14·4) = $\frac{24 \times 14·4}{25}$ = $\frac{345·6}{25}$ = 13·824 = the chord B C: and, by analogy or proportion, 13·824 : 14·4 : : 3 : 3·125. But, $\frac{360}{24}$ = 15 = an arc subtending the angle B A C, and therefore subtending the chord B C, when A B C denotes one of 24 equal isosceles triangles inscribed in a circle; and $\frac{24}{25}$(15) = 14·4. But, $\frac{14·4}{2}$ = 7·2, and, $\frac{15}{2}$ = 7·5 : and by analogy or proportion : 7·2 : 7·5 : : 3 : 3·125. Now, when A B C denotes one of 24 equal isosceles triangles inscribed in a circle, half the chord B C $= \frac{14{\cdot}4}{2} = 7{\cdot}2 =$ D C or D B. 8 (D B) = 8 × 7·2 = 57·6 = A B ; therefore, $AB^2 - DB^2 = 57{\cdot}6^2 - 7{\cdot}2^2 = 3317{\cdot}76 - 51{\cdot}4 = 3265{\cdot}94 = AD^2$; therefore, $\sqrt{3265{\cdot}94} = 57{\cdot}148228\ldots =$ A D : and the triangle A D B is an incommensurable right-angled triangle. But, $\frac{DB}{AB} = \frac{7{\cdot}2}{57{\cdot}6} = {\cdot}125$: and ·125 is the *trigonometrical* sine of the angle D A B. $\frac{AD}{AB} = \frac{57{\cdot}148228}{57{\cdot}6} = {\cdot}9921567$: and ·9921567 is the *trigonometrical* sine of the angle A B D ; and *trigonometrical* cosine of the angle D A B. Will you venture to tell me that the *trigonometrical* sines and cosines of angles are not the complements of each other ? Now, the logarithm corresponding to the natural number ·125 is 9·0969100, and this is the log-sin of the angle D A B. The logarithm corresponding to the natural number ·9921567 is 9·9968502 ; and this is the log-cos. of the angle D A B.

Now, Sir, conceive me to have given the three last paragraphs, in lieu of those on pages 10 and 11 of my Letter of the 23rd November! How would you have gone to work to demonstrate the fallacy of the proof, by Logarithms, on page 12, that the hypothenuse is to the base or shortest side in the ratio of 8 to 1, when the acute angle of the right-angled triangle A D B is an angle of 7° 12′? Will you venture to tell me, that the angle D A B is not an angle of 7° 12′, when the isosceles triangle A B C denotes one of 25 equal isosceles triangles inscribed in a circle ? Well, then, I have proved in my Letter of the 19th December, that half the arc contained by two radii of a circle, multiplied by radius, is equal to the area of that part of the circle contained by the two radii, and this fact is admitted by "*recognised Mathematicians.*" Hence: $25\left(\frac{{\cdot}125}{2}\right) = 24\left(\frac{3{\cdot}125 \div 24}{2}\right)$, that is, 25 (·0625) = 24 (·065104166666...) = 1·5625 = semi-circumference of a circle of diameter unity $= \frac{\pi}{2}$; and is equal to half the area of a circle of radius 1 : and it follows, that $25\left(\frac{14{\cdot}4}{2}\right) = 24\left(\frac{15}{2}\right)$; that is, 25 (7·2) = 24 (7 5) = 180 ; and is equal to the semi-circumference of a circle of radius 57·6 : and $3\frac{1}{8}(57{\cdot}6^2) = 3{\cdot}125 \times 3317{\cdot}76 = 10368$

= area of a circle of circumference = 360. Can YOU make the area of a circle of circumference 360 = 10368? Certainly not; and you know it! Well, then, the "*onus probandi*" rests with you to controvert these facts; and, as a matter of course, YOU, as a "*recognised Mathematician*," can controvert them, if they be not founded on the impregnable "*rock*" of *geometrical* truth, and beyond the possibility of being shaken by the "*waves*" of mathematical ingenuity.

With this communication, I shall send you a copy of the London *Correspondent* of February 3rd, 1866, in which you will find a Letter of mine, bearing distinctly upon the points I have now brought under your consideration. If truth be your object, as you profess it to be, you will give the Letter in that Journal your careful consideration: and I must direct your particular attention to the fact, that $3\left(\frac{6}{6\cdot25}\right) = 3\cdot125\left(\frac{6}{6\cdot25}\right)^2$, and that this equation or identity = 2·88, and is equal to the area of a circle of circumference = 6.

FROM THE "CORRESPONDENT," FEBRUARY 3, 1866.

QUADRATURE OF THE CIRCLE.

SIR,—I am afraid some of your readers will be heartily tired of circle-squaring, and had not a fresh champion of orthodoxy sprung up in the person of Mr. E. L. Garbett, I should not have troubled you with another "*slice of the Seaforth mince* π," without pausing for a time. Were I to remain silent, however, it would be taken for granted that his letter is unanswerable, and I must therefore beg of you to favour me with space in your next publication for a reply.

Well, then, I at once admit the correctness of Mr. Garbett's figures, but I dispute the argument he founds upon them. The chord of 15° = side of a regular polygon of 24 sides inscribed in a circle; and it is obvious that to whatever extent we may double the number of sides of polygons inscribed in a circle, as we increase the perimeters we increase the areas in like proportion, but can never make a polygon equal in perimeter and area to the circumference and area of its circumscribing circle. Now, if the radius of a circle = 1, the perimeter of an inscribed regular hexagon = 6, and the area of the hexagon = 2·598075. If to this be added the sum of the areas of the 12 right-angled triangles about the hexagon, which make up a dodecagon or 12-sided polygon = ·401925, we obtain the area of the dodecagon = 6 (radius × semi-radius) = 3. Now, if π denote the area of a circle equal in circumference to the perimeter of the hexagon; y the area of the dodecagon; and z the area of the circumscribing circle; $3 : \pi :: x : y$; and $3^2 : \pi^2 :: x : z$, whatever be the arithmetical value of π. But, $3\left(\frac{6}{6{\cdot}25}\right) = 3{\cdot}125\left(\frac{6}{6{\cdot}25}\right)^2$ and these equations = 2·88 = area of a circle of circumference = 6; therefore, (area of circle of circumference = 6 — area of hexagon of perimeter = 6) = 2·88 — 2·598075 = ·401925 = area of the 12 right-angled triangles about the hexagon. Thus, area of the circle, plus area of the triangles = 2·88 + ·401925 = 3·281925. This is an arithmetical quantity greatly in excess of the area of a circle of radius 1. But, if to the area of a circle equal in circumference to the perimeter of the dodecagon, or 12-sided polygon, we add the difference between the areas of the 12 and 24-sided polygons, we obtain a smaller arithmetical quantity than 3·281925; and by continuing this process we may still further reduce this quantity at every step, but when we have extended the calculations to the exhausting point, we have still a quantity in excess of the area of a circle of radius 1. Hence, the 47th proposition of the first book of Euclid is inapplicable (directly) to the measurement of a circle.

Within the last ten days I have received three letters from a Cornish gentleman, written in the most kindly spirit, in one of which the writer gently reproves both my opponents and myself for using hard words, which he truly says are "*useless and irritating.*" He observes :—"*When two disputants find they cannot agree in axioms or fundamentals, it is best to leave off disputing, which, if pursued, generally becomes mere reviling.*" I shall be glad if for the future all parties to the controversy on the ratio of diameter to circumference in a circle, will bear these facts in mind. The gentleman in question differs from me and adopts the same line of argument as Mr. Garbett, but puts it in a somewhat different form. He treats the subject, however, so distinctly and intelligibly as to leave no difficulty in dealing with it.

Let O A B represent one of 25 equal isosceles triangles inscribed in a circle. Let O denote the angle at the centre of the circle, A B its subtending chord, A C B its subtending arc, and O D a straight line bisecting the angle O, and its subtending chord A B. A diagram of this figure may readily be constructed by any geometrician.

The following is an illustration of my correspondent's argument and conclusion :—

The angle O $= \frac{360^\circ}{25} = 14^\circ\ 24'$

The angle A O D $= \frac{14^\circ\ 24'}{2} = 7^\circ\ 12'$

Therefore, A D = half the chord A B = A O sine $7^\circ\ 12'$ = sine $7^\circ\ 12'$ for A O radius = 1. The natural sine of $7^\circ\ 12'$ as per tables = ·125333 ; therefore, 2 (·125333) = ·250666 = the *(apparent)* value of the chord A B, and is greater than $\frac{3·125}{12·5} = ·25$; therefore my correspondent draws the conclusion that the chord A B is greater than its subtending arc A C B.

Now, this argument is, no doubt, extremely plausible, but I meet it in the following way :—

The circular measure of the angle O = arc A C B $= \frac{14^\circ\ 24' \times \pi}{180} = \frac{\pi}{12·5}$, whatever be the arithmetical value of π. Hence, 12·5 (arc A C B) = 12·5 × $14^\circ\ 24' = 180^\circ$, = semi-circumference of the circle ; 4 (arc A C B) = 4 × $14^\circ\ 24'$, = $57^\circ\ 36'$ = radius ; $\frac{12·5}{4} = 3.125 = \pi$; and 2π (radius) = $6·25 \times 57^\circ\ 36' = 360^\circ$ = circumference. Again : The perimeter of a regular inscribed hexagon to a circle of diameter unity = 6 times radius = 3 ; Hence, since the property of one circle is the property of all circles, and as π denotes the circumference of a circle of diameter unity ; it follows of necessity, that $\frac{3}{\pi}$ expresses the ratio between the perimeter of every regular hexagon and the circumference of its circumscribing circle. Thus, the angle O is to an angle of 15° as the perimeter of a regular hexagon to the circumference of its circumscribing circle ; that is to say, $3 : 3·125 :: 14^\circ\ 24' : 15^\circ$; therefore, 6 times radius = 6 × $57^\circ\ 36' = 345^\circ\ 36'$ = perimeter of a regular inscribed hexagon to a circle of 360° ; therefore, $\frac{\pi}{3}(345^\circ\ 36') = \frac{3·125 \times 345^\circ\ 36'}{3} = 360^\circ$ = circumference.

But further: The angles at the centre of a circle are to each other as their subtending arcs, therefore, the arc A C B is to an arc of 15° in the ratio of 3 to 3·125.

Now, the circular measure of an angle of 14° 24′ to a circle of radius 1 $= \frac{864' \times 3{\cdot}125}{180 \times 60} = {\cdot}25 =$ arc A C B; therefore, 12·5 (arc A C B) = 12·5 × ·25 = 3·125 = semi-circumference of a circle of radius 1; 4 times arc = 4 × ·25 = 1 = radius; and 2 π (radius) = 6·25 = circumference. We thus arrive at a similar conclusion, whether we make our calculations from a circle of circumference 360° or a circle of radius 1. But further: The circular measure of the arc A C B = semi radius of a circle of diameter unity; therefore, π times arc A C B = π times the square of the radius to a circle of diameter unity, $= \frac{\pi}{4} = 12{\cdot}5 \left(\frac{\pi}{50}\right) = 12{\cdot}5\,(s\,r^2) =$ area of a circle of diameter unity; and since the property of one circle is the property of all circles, it follows of necessity that $12{\cdot}5\,(s\,r^2) =$ area in every circle. My opponents may readily convince themselves that $\left\{12{\cdot}5 \left(\frac{\pi}{50}\right)\right\} =$ area of a circle of diameter unity, by means of any hypothetical value of π. These facts I challenge Mr. Garbett to controvert, and I ask him as an honest controversialist, to make the mysterious "*mince* π" 3·14159265, &c., harmonize with them. If he find this impossible, let him admit that he is checkmated.

Mr. Alex. Edw. Miller seems to have been sadly afraid of my seizing upon a typographical error and making a handle of it. I am not the man to play any such game. I had observed the error, but understood perfectly what he meant. If I had had the correction of the printer's mistakes, I might probably have employed the expression $\frac{1}{2}(\sqrt{2})$ instead of $\frac{1}{\sqrt{2}}$, the former being more readily worked out arithmetically.

In conclusion, I may observe: The question of whether the problem of the quadrature of the circle can or can not be solved, depends upon the possibility of ascertaining the true value of π. Notwithstanding the many letters of mine which have appeared in the *Correspondent* on this subject, Mr. Miller would appear to have been quite oblivious as to my object, for he observes: "As to discussing with him (Mr. Smith), or any one else, the question whether

π does or does not $= 3\frac{1}{8}$, it never entered into my head." This looks very like an attempt to wriggle out of a difficulty.

I am Sir, yours very respectfully,
JAMES SMITH.

I do not regret the unintentional lapsus in my Letter of the 23rd November, and I will tell you why. The original question in dispute between us, was the value of π. How many times have you charged me with evading the real question at issue, by attempting to change the subject? It never entered into your mathematical philosophy, to conceive of a difference between the *natural*, the *geometrical*, and the *trigonometrical* sine of an angle: and, but for my blunder, we might have been involved in a discussion on these points at a very early stage of our controversy. To avoid this I was necessitated to employ the term *natural sine* in the same sense that is attached to it by Mathematicians.

Well, then, ·125 is the *geometrical* sine of an angle of 7° 30′: but is not the *trigonometrical* sine of this angle but the *trigonometrical* sine of an angle of 7° 12′. I have proved in this and my last Letter—which should be taken together—that the ratio of chord to its subtending arc is *arithmetically* an invariable ratio, whatever may be the number of sides of a polygon inscribed in a circle. These facts ought to convince you of the fallacy of working round a curved line by the 47th proposition of Euclid's first book, in the search after π.

In conclusion: If you are a "*reasoning geometrical investigator*," you will cease to meet argument with assertion, and endeavour to convince yourself whether the matters I have brought under your notice are, or are not, true. If you have a "*logical mind*," which you *assume* you have,—and, by implication, *assume* that I have not—you will find that they are true: and, if you are "*a gentleman and a man of honour*," you will then take "*the earliest opportunity to withdraw*" the slanderous charges you brought against me in your original communication.

Yours faithfully,
JAMES SMITH.

THE REV. PROFESSOR WHITWORTH.

P.S.—I must ask you to be so kind as return me the copy of the *Correspondent*, when you have carefully perused the Letter referred to. Of course you need not be in a hurry about it. You will find some printer's errors in the Letter, but they are such as you will readily detect.

JAMES SMITH *to* THE REV. PROFESSOR WHITWORTH.

BARKELEY HOUSE, SEAFORTH,
18th, January, 1869.

SIR,

A week ago I posted a Letter to your address, and with it a Pamphlet and a copy of the London *Correspondent* of the 3rd February, 1866. As I have no acknowledgment of the receipt of the latter documents, it may be that they have not fallen into your hands. If this be so, please inform me, and oblige,

Yours faithfully,
JAMES SMITH.

JAMES SMITH *to* THE REV. PROFESSOR WHITWORTH.

BARKELEY HOUSE, SEAFORTH,
25th January, 1869,
Posted 1st February.

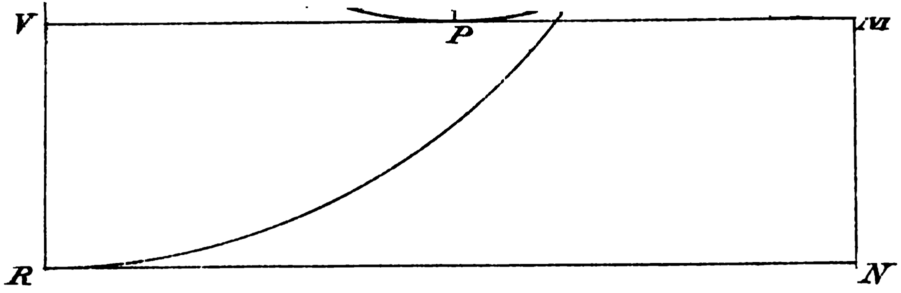
SIR,

In your Letter of the 30th November you said:—"*If you can give me a proof in which I see nothing illogical or unsound, I shall immediately publish it in the Mathematical Journal which I edit.*" I imposed no such obligation upon you, and did not even know that you were the Editor of "*The Oxford, Cambridge and Dublin Messenger of Mathematics*," until you gave me that piece of information: But, my good Sir, you have imposed a moral obligation upon yourself. I have already furnished you with many proofs, that the circumference of a circle of diameter unity is $\frac{25}{8} = 3{\cdot}125$, neither more nor less, and you have never attempted to prove that my demonstrations are either "*illogical or unsound.*" But, passing these by, I will now give you another proof—I have others in reserve—and the "*onus probandi*" will rest with you to make an attempt to demonstrate that my proof is "*illogical or unsound:*" and I do not *see* how you can escape the obligation to do this, as "*a gentleman, a man of honour,*" and a Christian minister, without doing *violence* to your own conscience. *You* must know, that self-imposed obligations are as binding upon conscience—although merely having reference to a *moral* or *scientific* subject—as the *pecuniary* obligation is binding upon the conscience of every honest man, to pay his just debts. You conclude your Letter of the 17th December, by observing:—"*You invite me, in your quotation from Solomon, to speak plainly in reproof of the course you have taken. But I prefer to reserve reproof for knaves and fools, and to meet ignorance with instruction.*" I have read your Letters in vain to find any attempt to meet *my* "ignorance" with *your* "instruction;" and I find it difficult to persuade myself, that you have *implicit faith* in your own *assertion.*

You may take the following as the construction of the geometrcal figure (Fig. 1.)represented by the enclosed diagram. (*See Diagram IX.*)

Let A B be a straight line. Produce A B to C, making A C equal to $\left\{\left(AB + \frac{AB}{4}\right) + \frac{1}{4}\left(AB + \frac{AB}{4}\right)\right\}$. Bisect A C at E, and with E as centre, and E A or E C as interval, describe the circle X. With B as centre and B A as interval, describe the circle Y: and produce A C, to meet and terminate in the circumference of the circle Y at the point D. From the point B, draw a straight line perpendicular to A D, to meet and terminate in the circumference of the circle X at the point F, and join F A, F E, and F C. With C as centre and C D as interval, describe an arc, to meet a tangent of the circle X drawn from C, at the point H. With B as centre and B C as interval, describe an arc, to meet F B produced at the point G. Produce F G to meet and terminate in the circumference of the circle Y, at the point P. With A as centre and A C as interval, describe an arc, to meet a tangent of the circle Y drawn from the

DIAGRAM IX.



point A, at the point R. From the point R, draw a straight line parallel to A D, to meet a tangent of the circle Y drawn from the point D, at the point N. From the points K, L, and M, draw straight lines through the points H, G, and P, to meet the line A R in the points S, T, and V. It is self-evident that all these lines are parallel to A D the diameter of the circle Y, and perpendicular to the line F P.

Now, let A B = 5. Then: $\left\{\left(AB + \frac{AB}{4}\right) + \frac{1}{4}\left(AB + \frac{AB}{4}\right)\right\} = \left\{\left(5 + \frac{5}{4}\right) + \frac{1}{4}\left(5 + \frac{5}{4}\right)\right\} = (6{\cdot}25 + 1{\cdot}5625) = 7{\cdot}8125 = AC$. But, $AD = 2(AB) = 10$, by construction; and it follows of necessity, that $(AD - AC) = (10 - 7{\cdot}8125) = 2{\cdot}1875 = CD$; and, $(AC - AB)$, or, $(BD - CD) = BC$: that is, $(7{\cdot}8125 - 5) = (5 - 2{\cdot}1875)$, and this equation or identity $= 2{\cdot}8125 = BC$. Hence: $(AB + BC + CD) = (5 + 2{\cdot}8125 + 2{\cdot}1875) = 10 = AD$. But, $\frac{3}{4}(AB) = \left(\frac{3 \times 5}{4}\right) = \frac{15}{4} = 3{\cdot}75 = FB$; and $\frac{5}{4}$(A B, $= \left(\frac{5 \times 5}{4}\right) = \frac{25}{4} = 6{\cdot}25 = FA$: $\frac{3}{4}(FB) = \left(\frac{3 \times 3{\cdot}75}{4}\right) = \left(\frac{11{\cdot}25}{4}\right) = 2{\cdot}8125 = BC$; and, $\frac{5}{4}(FB) = \left(\frac{5 \times 3{\cdot}75}{4}\right) = \left(\frac{18{\cdot}75}{4}\right) = 4{\cdot}6875 = FC$. You may readily convince yourself, that $(FC^2 - BC^2) = (FA^2 - BA^2)$ and this equation or identity $= FB^2$. Hence: By analogy or proportion, $AB : FB :: FB : BC$; that is, $5 : 3{\cdot}75 :: 3{\cdot}75 : 2{\cdot}8125$; and thus, "*on all shewings*" $BC = 2{\cdot}8125$, when $AD = 10$. Now, $\frac{AC}{2} = \frac{7{\cdot}8125}{2} = 3{\cdot}90625 = EC$ or EA; and it follows of necessity, that $(EC - BC) = (AB - EB)$, that is, $(3{\cdot}90625 - 2{\cdot}8125) = (5 - 3{\cdot}90625)$; and this equation or identity $= 1{\cdot}09375 = EB$. Hence: $EB = \frac{7}{24}(FB)$ or, $\frac{7}{25}(FE)$; that is, $\frac{7}{24}(3{\cdot}75) = \frac{7}{25}(3{\cdot}90625) = 1{\cdot}09375 = EB$. It is self-evident that $EF = EA$, for they are radii of the circle X; and it follows, that F B and B E the sides that contain the right angle in the triangle F B E, are in the ratio of 24 to 7. But, $\frac{AC}{2} = \frac{7{\cdot}8125}{2} = 3{\cdot}90625$, and it follows, that $(AE^2 - EB^2) = (FE^2 - EB^2)$, that is, $(3{\cdot}90625^2 - 1{\cdot}09375^2) = 15{\cdot}2587890625 - 1{\cdot}1962890625) = 14{\cdot}0625 = FB^2$; therefore, $\sqrt{14{\cdot}0625} = 3{\cdot}75 = FB$; and thus, "*on all shewings*," $FB = 3{\cdot}75$

when $AB = 5$. Hence: $\{FE^2 + EA^2 + 2(EA \times EB)\} = \{3{\cdot}90625^2 + 3{\cdot}90625^2 + 2(3{\cdot}90625 \times 1{\cdot}09375)\} = (15{\cdot}2587890625 + 15{\cdot}2587890625 \times 8{\cdot}544921875) = 39{\cdot}0625 = FA^2$; therefore, $\sqrt{39{\cdot}0625} = 6{\cdot}25 = FA$. You will observe that FA^2 is equal to 10 times F E or A E; or in other words, the area of a square on F A is equal to the area of a rectangle of which A D and A E are sides. By Euclid : Prop. 2 : Book 2 : "*If a straight line be divided into any two parts, the square of the whole line is equal to the sum of the rectangles contained by the whole line and each of its parts.*" Hence: $(AD \cdot AE + AD \cdot ED) = AD^2$, that is, $\{(10 \times 3{:}90625) + (10 \times 6{\cdot}09375)\} = (39{\cdot}0625 + 60{\cdot}9375) = 100 = AD^2$, and is equal to the area of a circumscribing square to the circle Y. Well then, $FA = 6{\cdot}25$, when $AB = 5$. But, $\frac{3}{4}(FA) = \frac{3 \times 6{\cdot}25}{4} = \frac{18{\cdot}75}{4} = 4{\cdot}6875 = FC$; and A F C is a right-angled triangle (Euclid: Prop. 31 : Book 3); therefore, $(FA^2 + FC^2) = (6{\cdot}25^2 + 4{\cdot}6875^2) = (39{\cdot}0625 + 21{\cdot}97265625) = 61{\cdot}03515625 = AC^2$; therefore, $\sqrt{61{\cdot}03515625} = 7{\cdot}8125 = AC$. Hence: The sides that contain the right angle in the triangle A F C, that is, the sides F C and F A are in the ratio of 3 to 4: the side F A is to the side A C in the ratio of 3 to 5. The triangles on each side of F B are similar to the whole triangle A F C, and it follows from Euclid: Prop. 2: Book 2: that $\{(AC \cdot AV) + (AC \,.\, BG)\} = AC^2$; that is, $\{(7{\cdot}8125 \times 5) + (7{\cdot}8125 \times 2{\cdot}8125)\} = (39{\cdot}0625 + 21{\cdot}97265625) = 61{\cdot}03515625 = AC^2$: or, in other words, the square of the line A C is equal to the sum of the rectangles contained by the whole A C and each of its parts A B and B C. So far Euclid would appear not to be at fault, either in the twelfth proposition of his second book, or the eighth proposition of his sixth book: but you cannot fail to perceive that in this geometrical figure, the lengths of all the straight lines contained in it, may be represented by commensurable quantities.

At this point you might say to me: I admit everything you have said so far, and the correctness of your computations: but the question at issue between us is the arithmetical value of the symbol π, which denotes the circumference of a circle of diameter unity; and what has either your facts or your geometrical figure to do with the circumference of a circle of diameter unity? On these points I will "meet" *your* "ignorance with instruction:" and I may tell you—even at the risk of being thought somewhat egotistical—that had our Newtons, La Places, and other early Geometers made the discoveries I have been privileged to make, π as a mathematical symbol would long ago have ceased to exist.

The following facts are incontrovertible, and will be *unhesitatingly* admitted by every honest "*recognised Mathematician.*" First: 2π (radius) = circumference in every circle. Second: $\pi (r^2)$ = area in every circle. Third: $\pi (r^2)$ = (circumference × semi-radius) and this equation or identity is equal to area in every circle. Fourth: The circumference of a circle of diameter unity, and the area of a circle of radius 1, are represented by the same arithmetical symbols, whatever be the value of π.

Now, with reference to the enclosed diagram, (Fig. 1,) we *know* that B A = B D, for they are radii of the circle Y: and we *know* that E A, E F, and E C are equal, for they are radii of the circle X, and we also *know* that the circumferences of circles are to each other as their radii. Well, then, are not the following fair and legitimate questions to put to a "*recognised Mathematician?*" Is it not supposed that the great value of Algebra over that "*indispensable instrument of science, Arithmetic,*" *is*, that it enables us to assume the symbol x to represent an unknown quantity—and by a process of algebraical reasoning, find the value of x? (This may be admitted to be true under certain circumstances, but I *deny* that it is a mathematical law of general and universal application. There are many circumstances, under which that "*indispensable instrument of science, Arithmetic,*" can be our only true guide or compass on our journey in search of scientific truth. The author of "*Choice and Chance,*"* who, it must be admitted is a "*recognised Arithmetician,*" can hardly be ignorant of this fact.) Well, then, is it possible from a given radius of the circles X and Y to find the values of their circumferences and areas with arithmetical exactness? Is it possible from a given circumference of the circles X and Y to find the values of the radii and areas of the circles with arithmetical exactness? To these questions I emphatically answer, Yes! and I may tell you, that there is nothing more simple, when we know how to furnish the proofs. But, is it possible to solve these theorems, without, at the same time, demonstrating the true arithmetical value of π? Certainly not! And the author of "*Choice and Chance*," can hardly be ignorant of this fact. Now, Sir, if your value of π be the true value, and you have "*implicit faith*" in it, you can furnish me with the solution of these theorems. If you decline to do so, you will force upon me the inference that, when in your original communication you said: "I *know* and *always teach* that the arithmetical value of π is a finite and determinate quantity:" you did so with a mental reservation, and was therefore a mathematical "*quibbler*" at the commencement of our controversy, that you are no better still, and (to employ your own words) "*should resign all claims to be a gentleman or a man of honour*."

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*The Rev Professor Whitworth.

Now, the author of "*Choice and Chance*" can have no difficulty in convincing himself, that $\frac{\pi}{4} = 12\frac{1}{2}\left(\frac{\pi}{50}\right)$, whatever be the value of π: and that this equation or identity represents the area of a circle of diameter unity. Well, then, I have proved that when A B a radius of the circle Y = 5; then, A F = 6·25 : F B = 3·75 : and A E and F E = 3·90625. But, when A B = 1, then, A F = 1·25 : F B = ·75 : and A E and F E = ·78125. Now, by analogy or proportion, 1·25 : 3·90625 : : 1 : 3·125, and it follows, that $\frac{1 \cdot 25}{3 \cdot 90625}$ and $\frac{1}{3 \cdot 125}$ are equivalent ratios: and I shall now proceed to prove that both express the true ratio between diameter and circumference in every circle.

Let A B the radius of the circle Y = 5. Then: A F = 6·25. Hence: $\left\{(AF\,.\,AB) \times \frac{AB}{2}\right\} = \frac{AF}{2}(AB^2)$; that is, $\left\{(6 \cdot 25 \times 5) \times 2 \cdot 5\right\} = \frac{6 \cdot 25}{2}(5^2)$, or, $(31 \cdot 25 \times 2 \cdot 5) = 3 \cdot 125 \times 25 = 78 \cdot 125$. Again: $\left\{3 \cdot 125 (AD) \times \frac{AD}{4}\right\} = (31 \cdot 25 \times 2 \cdot 5) = 78 \cdot 125$. Again: $3 \cdot 125 (AB^2) = (AD \times AR)$; that is, $(3 \cdot 125 \times 25) = (10 \times 7 \cdot 8125) = 78 \cdot 125$. Again: $(AF^2 + FB^2 + AB^2) = \{(AD \times AB) + (AD \times BC)\}$, that is, the sum of the squares of the three sides of the right-angled triangle A F C, is equal to the sum of the rectangles A D . A V and A D . A T: that is, $(6 \cdot 25^2 + 3 \cdot 75^2 + 5^2) = \{(10 \times 5) + (10 \times 2 \cdot 8125)\} = 78 \cdot 125$: and it follows, that the sum of the squares of the three sides of the right-angled triangle A F C, is equal to the sum of the areas of the rectangles A V M D and A T L D. Again: $AD^2 = 10^2 = 100$ = area of a circumscribing square to the circle Y, and half the area of a circumscribing square = area of an inscribed square to every circle ; therefore, $\frac{100}{2} = 50$ = area of an inscribed square to the circle Y. But, (A D × A V) = 10 × 5 = 50 : and it follows, that the rectangle A V M D and an inscribed square to the circle Y are exactly equal in superficial area ; and when A B = 5 = 50. Hence : $\left\{\left(50 + \frac{50}{4}\right) + \frac{1}{4}\left(50 + \frac{50}{4}\right)\right\} = \left(10 \text{ A F} \times \frac{10 \text{ A F}}{4}\right) = (62{\cdot}5 + 15{\cdot}625) = 78{\cdot}125$.

Now, $\frac{3{\cdot}125}{4} = 12{\cdot}5\left(\frac{3{\cdot}125}{50}\right) = \frac{78{\cdot}125}{100} = {\cdot}78125$: and it is incontrovertible, that the area of a circle of radius 1 is represented by the same arithmetical symbols as the circumference of a circle of diameter unity: and the area of a circle of radius 1, equal to four times the area of a circle of diameter unity.

Now, let A D the diameter of the circle Y = 1. I have proved that in the right-angled triangle F B A, the sides F B and B A which contain the right angle, are in the ratio of 3 to 4, by construction. Hence: When A D = 1, then, $\text{A B} = \frac{1}{2} = {\cdot}5 : \frac{3}{4}(\text{A B}) = \frac{3 \times {\cdot}5}{4} = \frac{1{\cdot}5}{4} = {\cdot}375$ = F B: and, $\frac{5}{4}(\text{A B}) = \frac{5 \times {\cdot}5}{4} = \frac{2{\cdot}5}{4} = {\cdot}625$ = F A; therefore, $(\text{A B}^2 + \text{F B}^2 + \text{F A}^2) = ({\cdot}25 + {\cdot}140625 + {\cdot}390625) = {\cdot}78125$, and is equal to $3{\cdot}125\,(\text{A B}^2)$ when A D = 1 ; and is exactly equal to $\frac{3{\cdot}125\,(\text{A B}^2)}{100}$, when A D = 10: that is, $\frac{3{\cdot}125 \times 5^2}{100} = (3{\cdot}125 \times {\cdot}5^2)$, or, $\frac{3{\cdot}125 \times 25}{100} = 3{\cdot}125 \times {\cdot}25 = {\cdot}78125$. Will you dare to tell me, that when A D the diameter of the circle Y = 10, the area of the circle Y is not equal to 100 times the area of a circle of diameter unity, whatever be the value of π? I trow not ! Well, then, have I not given you a series of identities in connection with the enclosed diagram, (Fig. 1,) all *tending*—(not to an infinite series)—to a "*finite and determinate*" value of the circumference of a circle of diameter unity, or the area of a circle of radius 1 ? At this mode of expression you could, and probably would, raise a "*quibble*." But, suppose me to put the question :—Have I not given you a number of identities in connection with the enclosed diagram, (Fig. 1,) all proving, beyond the possibility of dispute or cavil by any honest "*recognised Mathematician*," that the area of a circle of diameter unity, is equal to one-hundredth part of the area of a circle of diameter 10? How would you raise a quibble? Again: Let A D the diameter of the circle Y = 2. Then: A B a radius of the circle = $\frac{2}{2} = 1$; and the author of "*Choice and Chance*" may readily convince himself, that when A B = 1, the sum of the squares of the three sides of the right-angled triangle F B A = 3·125, and is exactly equal to $4\,(AB^2 + FB^2 + FA^2)$ when A D = 1: and since by no other value of π, either greater or less than $\frac{25}{8} = 3{\cdot}125$, can we make $\pi\,(AB^2) = (AB^2 + FB^2 + FA^2)$ whatever be the value of A B; it follows of necessity, that $\frac{25}{8}$ expresses the true ratio of circumference to diameter in every circle, and makes $\frac{25}{8} = 3{\cdot}125$, the true value of π, making 8 circumferences of every circle exactly equal to 25 diameters. Hence: $3{\cdot}125\,(AB^2) = (AD \times AR)$, and it follows, that the circle Y and the rectangle A R N D are exactly equal in superficial area. But, the area of the rectangle A R N D is equal to the sum of the areas of the rectangles A V M D and A T L D = $3{\cdot}125\,(AB^2)$; and it follows of necessity, that the sum of the areas of the rectangles A V M D and A T L D, are exactly equal to the area of the circle Y. Now, I have proved that the area of the rectangle A V M D is exactly equal to the area of an inscribed square to the circle Y, and it follows of necessity, that the area of the rectangle A T L D is exactly equal to the difference between the area of the circle Y and the area of its inscribed square: and the area of the rectangle A S K D exactly equal to the difference between the area of the circle Y and the area of its circumscribing square. From these incontrovertible facts, it follows of necessity, that the sum of the areas of the rectangles A S K D, A T L D, and A V M D, is exactly equal to the sum of the areas of the rectangles A S K D and A R N D; and this equation or identity is exactly equal to the area of a circumscribing square to the circle Y.

So far I have hardly made any use of the circle X, and yet, it plays a very important part in the question at issue between us; that is to say, in demonstrating the true arithmetical value of π; or, in other words, in demonstrating the true arithmetical value of the circumference and area of a circle of diameter unity: and this I shall now proceed to prove.

I have already proved, that when A B the radius of the circle Y = 5, then, A F = 6·25, and A E a radius of the circle X = 3·90625. Now, A E × A F = 3·90625 × 6·25 = 24·4140625 : and $\frac{AE}{2} = \frac{3·90625}{2}$ = 1·953125 is the semi-radius of the circle X. But, 24·4140625 × 1·953125 = 3·125 $(3·90625^2)$; that is, 24·4140635 × 1·953125 = 3·125 × 15·2587890625 ; and this equation or identity = 47·6837158203125 = area of the circle X : and it follows, that 24·4140625 is the circumference of the circle X, and is equal to $10\left(\frac{AF}{4}\right)^2$. Proof: $\left(\frac{AF}{4}\right)^2 = \left(\frac{6·25}{4}\right)^2 = 1·5625^2 = 2·44140625$; therefore, 10 (2·44140625) = 24·4140625 = the circumference of the circle X, and $\frac{24·4140625}{3·125}$ = 2 (3·90625) = 7·8125 = the diameter of the circle X ; when A B the radius of the circle Y = 5. Again: The radius of the circle X : the radius of the circle Y : : the area of the circle X : the area of a circumscribing square to the circle X : that is, 3·90625 : 5 : : 47·6837158203125 : 61·03515625. Proof: 2 (3·90625) = 7·8125 = the diameter of the circle X ; and $7·8125^2$ = 61·03515625 : and it follows, that 61·03515625 is the area of a circumscribing square to the circle X. Now, I have more than once proved in previous Letters, that $\frac{·78125}{1}$ and $\frac{1}{1·28}$ are equivalent ratios ; and it follows, that the quotient of any arithmetical quantity divided by ·78125, is equal to the product of the same quantity multiplied by 1·28. But, $\frac{3·125}{4}$ = ·78125 : and $\frac{61·03515625}{·78125}$ = 61·03515625 × 1·28 = 78·125 : and I have proved that the area of the circle Y = 78·125 when A D the diameter = 10. Hence : If the area of the circle Y = 78·125, and denote the area of a square, the area of a circumscribing square to the circle X, equal to 61·03515625, will be the area of an inscribed circle.

These facts are quite unique ; that is to say, we cannot divide the line A B the diameter of the circle Y in any other proportions then those into which it is divided in the enclosed diagram, and produce the foregoing results. Mathematicians may carp and "*quibble*," and tell me that "*practical geometry can prove nothing*;" but no Mathematician in the world can controvert the following conclusions. The equivalent ratios, $\frac{\cdot78125}{1}$, $\frac{1}{1\cdot28}$, $\frac{1\cdot25}{3\cdot90625}$, and $\frac{1}{3\cdot125}$, all express the true ratio of diameter to circumference in every circle, which makes 3·125 the circumference of a circle of diameter unity, and the area of a circle of radius 1 ; making $\frac{3\cdot125}{4} = \cdot78125$, the area of a circle of diameter unity, and equal to one-hundredth part of the area of a circle of diameter 10.

In your Letter of the 4th December, you said :—"*I suppose I am to infer, from your attempts to change the subject, that you have nothing further to adduce in support of* $\pi = 3\cdot125$; *and that my strictures on your quasi-proofs are unanswerable.*" Let it be assumed that I have not hitherto given you a proof, or anything deserving the name even of a "*quasi-proof*," that $\pi = 3\cdot125$. I am prepared to rest my Geometrical and Maththematical reputation upon the proofs I have already given you in this communication: and I set up a claim to the character of a "*gentleman*" and "*a man of honour.*" Well, then, pray try your hand again ; and if you succeed in proving my data and reasonings "*illogical or unsound,*" and so vitiate my conclusions, you shall not find me hesitate to admit, that I am wrong in setting up any claim to be thought either a Geometer or a Mathematician.

In your second Letter of the 28th November (referring to mine of the 23rd) you observe :—"*However, none of your results are very startling till we get to page* 21, *where you assume, without proof, (and I should say wrongly assume,) that certain acute angles in your figure are* 16° 16′ *exactly. Pray how do you arrive at this conclusion? Is not the angle a very small fraction less than this? I will grant that the sines of your angles are* ·28 *and* ·96, *but I deny that the angles are* 16° 16′ *and* 73° 44′." At this time, your communications came upon me thick and two-fold, and there was no keeping pace with you : but your reticence, latterly, enables me to revert to your enquiries, arguments, and assertions, and I shall now proceed to show you how I arrive at the conclusion that the angles you refer to are angles of 16° 16′ and 73° 44′.

Well, then, the triangles D F O, A D P, A F M, and D G H, in the geometrical figure represented by the diagram enclosed in my Letter of the 23rd November *(see Diagram I.)*, are similar right-angled triangles, and all similar to the right-angled triangle F B E, in the geometrical figure represented by the enclosed diagram, (Fig. 1.) The triangles O A H, C A L, D G L, P H L, P H A, A O P, A D G, D F A, A D L, A D C, and A K C, in the former diagram, are similar right-angled triangles, and all similar to the right-angled triangles F B A, and F B C in the latter diagram. But, F E A, a part of the right-angled triangle F B A, is an oblique-angled triangle; and if Euclid be not at fault in the 32nd proposition of his first book, the angles of the triangles F B E and F E A are together equal to four right angles: and I have proved that, according to Euclid: Prop. 12: Book 2. $\{FE^2 + EA^2 + 2\,(EA \times EB)\} = FA^2$. But, I have also proved, that when A B = 5, then, F A = 6·25: E A = F E = 3·90625: F B = 3·75: and E B = 1·09375.

Now, referring to the enclosed diagram, (Fig. 1), $\frac{EB}{FE} = \frac{1{\cdot}09375}{3{\cdot}90625} = {\cdot}28$, is the trigonometrical sine of the angle E F B. (Called by Mathematicians the natural sine of the angle.) $\frac{EB}{FE} = \frac{3{\cdot}75}{3{\cdot}90625} = {\cdot}96$, is the trigonometrical co-sine of the angle E F B. (Called by Mathematicians the natural co-sine of the angle.) The Logarithm corresponding to the natural number ·28 is 9·4471580, and this is the trigonometrical Log.-sine of the angle E F B: and the Logarithm corresponding to the natural number ·96 is 9·9822712, and this is the trigonometrical Log.-cosine of the angle E F B: and you will not dare to dispute that the sine and co-sine of any angle are the complements of each other.

Well, then, let the length of F E, the side subtending the right angle in the right-angled triangle F B E, be represented by any finite arithmetical quantity, say 77, and be given to find the lengths of the other two sides F B and B E, and prove that they are in the ratio of 24 to 7.

Then:

As Sin. of angle B = Sin. 90°	Log.	10·0000000
: the given side F E = 77	Log.	1·8864907
: : Sin. of angle E B F = Sin. 16° 16′	Log.	9·4471580
		11·3336487
: the required side B E		10·0000000
= 77 × ·28 = 21·56	Log.	1·3336487

Again:

As Sin. of angle B = Sin. 90°............................Log. 10·0000000
: the given side F E = 77..................................Log. 1·8864907
: : Sin. of angle F E B = Sin. 73° 44′.....................Log. 9·9822712

11·8687619
: the required side F B.................................... 10·0000000
= 77 × ·96 = 73·92 ....Log. 1·8687619

Hence:

B E : F B : : 7 : 24; that is, 21·56 : 73·92 : 7 : 24.

$\therefore \sqrt{(21{\cdot}56^2 + 73{\cdot}92^2)} = \sqrt{(464{\cdot}8336 + 5464{\cdot}1664)} = \sqrt{5929} = 77$ = the given side F E.

But further:

B E : F E : : 7 : 25; that is, 21·56 : 77 : : 7 : 25 : and, F B : F E : : 24 : 25; that is, 73·92 : 77 : : 24 : 25.

By Hutton's Tables:

As Sin. of angle B = Sin. 90°............................Log. 10·0000000
: the given side F E = 77..................................Log. 1·8864907
: : Sin. of angle E B F = Sin. 16° 16′.....................Log. 9·4473259

11·3338166
: the required side B E .... 10·0000000
= 21·568 approximately ....Log. 1·3338166

Again:

As Sin. of angle B = Sin. 90° ....Log. 10·0000000
: the given side F E = 77 ....Log. 1·8864907
: : Sin. of angle F E B = Sin. 73° 44′......... ....Log. 9·9822569

11·8687476
: the required side F B .... 10·0000000
= 73·917 approximately ....Log. 1·8687476

Again referring to the enclosed diagram (Fig. 1.) $\frac{FB}{FA} = \frac{3{\cdot}75}{6{\cdot}25}$ = ·6, is the trigonometrical sine of the angle F A B: and $\frac{BA}{EA} = \frac{5}{6{\cdot}25}$ = ·8, is the trigonometrical co-sine of the angle F A B. The Logarithm corresponding to the natural number ·6 is 9·7781513, and this is the trigonometrical Log-sin of the angle F A B: and the

Logarithm corresponding to the natural number ·8 is 9·9030900, and this is the trigonometrical Log-cosine of the angle F A B.

Well, then, let the length of F A, the side subtending the right angle in the right-angle triangle F A B, be represented by any finite arithmetical quantity, say 666, and be given to find the length of the other two sides F B and A B, and prove that they are in the ratio of 3 to 4.

Then:

As Sin. of angle B = Sin. of 90°	Log.	10·0000000
: the given side F A = 666	Log.	2·8234742
: : Sin. of angle F A B = 36° 52′	Log.	9·7781510
		12·6016255
: the required side F B		10·0000000
= 666 × ·6 = 399·6	Log.	2·6016255

Again:

As Sin. of angle B = Sin. of 90°	Log.	10·0000000
: the given side F A = 666	Log	2·8234742
: : Sin. of angle A F B = Sin. of 53° 8′	Log.	9·9030900
		12·7265642
: the required side A B		10.0000000
= 666 × ·8 = 532·8	Log.	2·7265642

Hence:

F B : A B : : 3 : 4; that is, 399·6 : 532·8 : : 3 : 4.

$\therefore \sqrt{(399{\cdot}9^2 + 532{\cdot}8^2)} = \sqrt{(159680{\cdot}16 + 283875{\cdot}84)} = \sqrt{443556}$ = 666 = the given side F A.

But further:

F B : F A : : 3 : 5; that is, 399·6 : 666 : : 3 : 5, and, A B : F A : : 4 : 5; that is, 532·8 : 666 : : 4 : 5.

By Hutton's Tables.

As Sin. of angle B = Sin. of 90°	Log.	10·0000000
: the given side F A = 666	Log.	2·8234742
: : Sin. of angle F A B = Sin. of 36° 52′	Log.	9·7781186
		12·6015928
: the required side F B		10·0000000
= 396·57 approximately	Log.	2·6015928

Again:

As Sin. of angle B = Sin of 90°……………………Log. 10·0000000
: the given side E A = 666 …………………………Log. 2·8234742
: : Sin. of angle A F B = Sin. 53° 8′ ………………Log. 9·9031084

12·7265826
: the required side A B ……………………… 10·0000000

= 532·82 approximately ……………………………Log. 2·7265826

Now, I make the sine and co-sine of the angle E F B, in the triangle F B E, to be ·28 and ·96—and on this point we are agreed—and $\cdot28^2 + \cdot96^2 = \cdot0784 + \cdot9216 = 1$. I make the sine and co-sine of the angle B A F, in the triangle F B A, to be ·6 and ·8, and $\cdot6^2 + \cdot8^2 = \cdot36 + \cdot64 = 1$. Both meet the requirement of the trigonometrical axiom Sin.^{2} + Cos.^{2} = unity in every right-angled triangle. Will you dispute this axiom in Trigonometry? Well, then, by computations based on the Logarithms of these numbers, I find the values of the sides that contain the right angle, from a given value of the sides that subtend it. Will you dare to tell me that my method of finding these values is "*illogical or unsound?*" You will observe that I make the ratio between the sides that contain the right angle, and the ratio of sine to co-sine, the same in either case, and both in harmony with the known and indisputable ratios of side to side, by construction. By the computations made from Hutton's Tables, the sides that contain the right angle, in the triangle F B E, are 21·568… and 73·917…, when the side that subtends the right angle is 77: and, 7 : 24 : : 21·568 : 73·947. This would make the angle E F B not—as you put it—less, but greater than an angle of 16° 16′. Again: By the computations made from Hutton's Tables, the sides that contain the right angle in the triangle F B A, are 399·57 and 532·82, when the side that subtends the right angle is 666. This would make the angle F A B less than an angle of 36° 52′. In both cases, the ***known and indisputable*** ratios of side to side, by construction, are destroyed. Well, then, the angles E A F and E F A are angles of 36° 52′: F E A is an angle of 106° 16′: E F B is an angle of 16° 16′: F E B is an angle of 73° 44′: and B is a right angle = 90°. Hence: the sum of all these angles is equal to four right angles.

Now, Sir, we are told by De Morgan, that "*Mathematics and logic are the two eyes of exact science.*" If this be so, *you*, as a "*recognised Mathematician*," and an admitted logician, can surely find the values of the angles in the triangles F B E and F B A, expressed in degrees: and if you dispute my value of them, are you not bound by *conscience*, as *a gentleman*, *man of honour*, and a fair controversialist, to prove my data to be *unsound*, my reasonings *illogical*, and my conclusions *fallacious?* To answer by vague generalities is not argument, or the way to meet any man's "*ignorance with instruction.*"

In my Letter to the Duke of Buccleuch, you will find a copy of a communication to De Morgan, in which I have shewn that, in a right-angled triangle of which the sides are 49, 1200, and 1201; or, in other words, that in a *commensurable* right-angled triangle derived from the consecutive numbers 24 and 25, the acute angle is an angle of 14 minutes. But, I have done more. In that communication I have shewn that, by computations from Hutton's Tables, we should make the hypothenuse and longer of the two sides that contain the right angle, equal within one-tenth part of a mile in 6000, when the shortest side is upwards of 24 miles in length. If this is not a *reductio ad absurdum* demonstration, that our Mathematical Tables of Sines, Co-sines, &c. are fallacious, what is, or can be?

I had written so far, when on calling at the office of the Lancashire Insurance Co., on Wednesday,—where I had not been for a week—I found a packet enclosing my last three Letters to you awaiting me, *apparently* unopened. I presume you wish me to believe that you returned them in the same state they left me, in fulfilment of the declaration made in your Letter of the 28th December. On enquiry, I found they had been left on the previous Friday, so that the Letter posted to you on the 4th January, must have been in your possession at least 14 days. How happens it that you did not return this Letter immediately? It may be that you were from home, and consequently, that the three Letters only came into your hands on the 19th January. If this be so, you will of course know it. To me, however, there is "*a dark mystery*"* involved, since I have an impression that a term commenced at Queen's College, on or about the 7th January. Here I must leave matters: your own *conscience* can unravel the *dark mystery* better than I can.

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* In the Correspondent there appeared a series of Letters headed "A DARK MYSTERY," in which Professor de Morgan and Mr. James Smith were made to play a conspicuous part. Mr. James Smith cannot say who was the writer—it could not be the learned Professor—but he has some reason for suspecting that he was, and is, one of the WE of the *Athenæum*.

In a foot-note on page 57 of my Letter to the Duke of Buccleuch, I have given a problem for solution. No Mathematician, whether "*recognised*" or otherwise, has ever solved it. I gave the solution to your "*College chum*," Mr. J. M. Wilson, in a long Letter, dated 9th November. It was my intention to have given you a copy of this Letter, but this is not now necessary, since you have resolved, like Mr. Wilson, not to read my communications: and probably now think as he does, that you "*made a mistake in corresponding with me*." If my health be spared, you will have an opportunity of seeing the Letter in print.*

In your second Letter of the 28th November, in reply to mine of the 23rd, you say: "*Finally, with reference to the paragraph at the foot of page 25, I must observe that as Euclid's reasoning is perfectly general, his results in the propositions you name are true universally, unless there be a flaw in his reasoning. If there is a flaw, you can doubtless point it out. If the propositions are not true universally, they are not true as Euclid states them, and therefore Euclid is at fault.*"

Now, omitting all reference to the construction of the geometrical figure represented by the enclosed diagram, Fig. 2, *(see Diagram X.,)* beyond the fact that A B is a diameter of the circle X and bisects the line G F, it may be admitted, that by the reasoning of Euclid in his theorem: Prop. 35: Book 3: A C × C B is equal to a square on C F; and that, *apparently*, there is no flaw in Euclid's reasoning; and yet, Euclid is at fault after all; but, his fallacy can only be detected, by means of applied mathematics to practical or constructive geometry. In your Letter of the 11th December, after certain "*strictures*" on a paragraph in mine of the 23rd November, you "*twit*" me on my ignorance of the theorem of Euclid: Prop. 35: Book 3: by putting the following questions:—"*Is it not rather wonderful that the author of 'Euclid at Fault' should be thus ignorant of the contents of the book he criticises? What will the public think of this, the next time you publish a pamphlet of correspondence?*"

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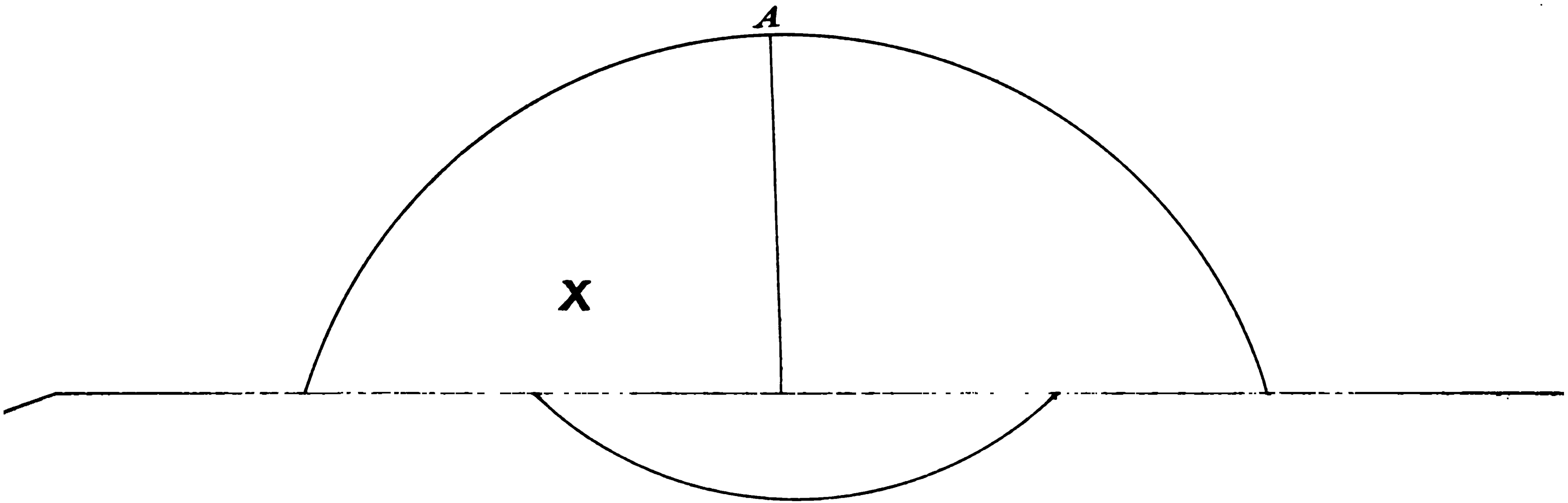
* This Letter will be found in Appendix C.

DIAGRAM X.



The following may be taken as the construction of the geometrical figure represented by Diagram X.

Let A B be a straight line divided into two unequal parts at C, so that A C shall be to C B in the ratio of 5 to 3. Bisect A B at O, and with O as centre and O A or O B as interval, describe the circle X. With B as centre and B C as interval, describe the circle Y. From the point B draw a straight line at right angles to A B and therefore tangental to the circle X, to meet and terminate in the circumference of the circle Y, at the point D. Produce B D to E, making D E equal to $\frac{7}{24}$ (B D). From the point E raise a perpendicular to B E, to meet a straight line drawn from the point C parallel to B E in the circumference of the circle X, at the point F, and join O D, F D, and F B. Produce F C, to meet and terminate in the circumference of the circle X, at the point G. It is self-evident, that C F and C G are tangental to the circle Y.

Let A C $= 5$. Then: B D $=$ C B $= 3$: D E $= \frac{7}{24}$ (B D) $= \frac{7 \times 3}{24} = \frac{21}{24} = {\cdot}875$; therefore, B E $= 3{\cdot}875$.

Now, by Euclid : Prop. 2 : Book 2 : $BE^2 = \{(BE \cdot BD) + BE \cdot DE)\}$; that is, $3{\cdot}875^2 = \{(3{\cdot}875 \times 3) + (3{\cdot}875 \times {\cdot}875)\}$; or, $(11{\cdot}625 + 3{\cdot}390625) = 15{\cdot}015625 = BE^2$; and is equal to a square on B E. But, by Euclid : Prop. 35 : Book 3 : a square on C F $=$ (A C $\times$ C B); that is, $CF^2 = (5 \times 3) = 15$. But, C F $=$ B E, for they are opposite sides of the parallelogram C B E F: and $BE^2 = 15{\cdot}015625$. Euclid no where proves this, nor could he, without the aid of applied Mathematics. Well, then, can squares on equal lines be unequal? Certainly not, and you *know* it! Hence: If Euclid be right in the Theorem : Prop. 35 : Book 3 : he is at fault in the Theorem : Prop. 2 : Book 2 : and conversely, if Euclid be right in the Theorem : Prop. 2 : Book 2 : he is wrong in the Theorem Prop. 35 : Book 3. He cannot be mathematically right in both, unless indeed, that "*indispensable instrument of science, Arithmetic*," upon which all mathematics are founded, be a mere "*mockery, delusion, and a snare.*" My Letter of the 29th December ought to have carried conviction to your mind, and I venture to tell you would have done so, had you read it, and been "*a reasoning geometrical investigator*," and to it I refer you as my method of meeting "*ignorance with instruction.*" It is not one of the Letters returned, and I presume you have it in your possession.

Now, Sir, the "*onus probandi*" rests with you to prove that Euclid is not at fault, and if he is not at fault either in Prop. 35 : Book 3 : or, Prop. 12 : Book 2 : *you*, Sir, as a "*recognised Mathematician,*" and admitted Arithmetician, can make $\{FD^2 + DB^2 + 2(DB \times DE)\} = FB^2$. How will you set about it? If you attempt to prove it by that "*indispensable instrument of science, Arithmetic,*" you will not only make Euclid "*upset*" himself, but you will find that you "*upset*" yourself at the same time, and so "*bring down two birds at one shot.*"

When I published the pamphlet "*Euclid at Fault,*" I had never heard of Mr. J. M. Wilson, or his treatise on "*Elementary Geometry:*" but I was subsequently induced to get it. To me it has been very suggestive, and has enabled me to detect some of my own blunderings, and led me to many discoveries in constructive geometry. Mr. Wilson observes, in his preface :—"*Everybody recollects, even if he have not daily experience, how unavailable for problems a boy's knowledge of Euclid generally is. Yet this is the true test of geometrical knowledge; and problems and original work ought to occupy a much larger share of a boy's time than they do at present.*" I have already adverted to De Morgan's sarcastic reference to the term "*quantity of turning*" introduced by Mr. Wilson in his sixth definition. You can hardly fail to have observed the importance Mr. Wilson attaches to parallels, and yet, he has not himself discovered the effect of deviation from parallelism in practical or constructive geometry.

Now, with reference to the geometrical figure represented by the diagram in "*Euclid at Fault,*" it is self-evident, from mere inspection, that there is not a straight line in the figure, parallel to A B the generating line of the diagram : and I have proved in my Letter of the 29th December, that if a "*quantity of turning*" be applied to a straight line in the direction from B to H, starting from the "*initial position*" K B, it will become parallel to O T, the hypothenuse of the right-angled triangle O B T, before it reaches the point H ; and it follows of necessity, that O B T and K B M are not similar right-angled triangles. I had carefully studied the properties of the geometrical figure represented by the diagram enclosed in my Letter of the 23rd November ; but, not knowing the effect of deviation from the parallel or perpendicular of the generating line, in the construction of a problem, I was inadvertently and unwittingly led into the blundering assumption, that O B T and K B M were similar right-angled triangles; and, consequently, that the parallelograms K L B H and K A D C, in the two diagrams, were similar parallelograms; and so, *erroneously*, making K H = $\frac{1}{4}$(K B) the diameter of the circle, in the diagram in "*Euclid at Fault.*" But, strange to say, the Mathematicians who have attacked that Pamphlet, and there are about a score,—*some of them recognised and known to fame*—have all failed to perceive my blunder, and have taken for granted, and reasoned accordingly, that OBT and K B M, in the diagram in "*Euclid at Fault,*" are similar right-angled triangles. Even De Morgan, who was quick enough to detect a flaw in the body of the Pamphlet, failed to discover the absurdity of my assumption with reference to the diagram. (See: Critique on "*Euclid at Fault*:" *Athenæum*, July 25, 1868.)

Now, Sir, will you compare the diagram in "*Euclid at Fault*" with the enclosed diagram, Fig. 2 *(see Diagram X)*. You will observe that, in the latter figure, C F and B E are perpendicular to A B the generating line of the diagram: while, in the former figure, B P and F H are neither perpendicular nor parallel to A B the generating line of the diagram. But, since A C, in the former, = K F, in the latter figure, and C B, in the former, = F B, in the latter figure, by construction, it follows of necessity, that the triangles O B D and O B T, in these geometrical figures, are similar and equal right-angled triangles, if the diagrams be drawn to the same scale: the triangles F E D and H P T are similar and equal right-angled triangles: and, the parallelograms C B E F and F B P H are similar and equal parallelograms. Well, then, can you, Sir, a "*recognised Mathematician,*" fail to perceive that, had I known when I published the Pamphlet "*Euclid at Fault,*" what Mr. J. M. Wilson has since been the "*instrument*" of forcing upon my attention, I might have demonstrated Euclid to be at fault just as well by one of these geometrical figures as the other, since the fact is incontrovertible that T P is as certainly = $\frac{7}{24}$ (B T), in the diagram in "*Euclid at Fault,*" as D E = $\frac{7}{24}$ (B D) in the enclosed diagram, Fig. 2?

It is hardly to be wondered at, that in adopting methods never before thought of by Mathematicians, in my search after π; or in other words, in my endeavours to find the true circumference of a circle of diameter unity, I should have occasionally fallen into blunders ; but I can conscientiously say, that I have never discovered a lapsus, or had one pointed out to me by an opponent, without at once frankly admitting it. Is it not more to be wondered at, that Mathematicians of the celebrity of Mr. J. M. Wilson and yourself, should have failed to discover, that the triangles O B T and K B M in the diagram in "*Euclid at Fault*," are not similar right-angled triangles? Some of my correspondents would have it that B F and B T are unequal, although they are radii of the arc F T: and one of them would have it that B F is greater than $\frac{3}{4}$(O B); but, as I have already said, not one of them discovered the real fallacy. Even Mr. J. M. Wilson, although he admits that the "*true test of geometrical knowledge*" is the construction of geometrical problems, would appear to be incompetent to apply his own theory.*

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* I sent a copy of the pamphlet, "*Euclid at Fault*," to Professor Harley, one of the Honorary Secretaries to Section A of the *British Association*. The Professor handed over the pamphlet to a young friend, just fresh from College, Mr. Henry W. Toller, of Stoney-gate House, Leicester. Mr. Toller addressed a Letter to me in which he argued, (and by a certain use of Euclid, *apparently* with some shew of reason) that B T is not equal to B F, although they are radii of the arc F T. In two courteous communications I pointed out to Mr. Toller the fallacy and inconsistency of his arguments, when I received a Letter from him in which the following expressions occurred:—"*Your Mathematical genius is but in its swaddling clothes.*" "*You must therefore be treated as a child.*" "*My little dear.*" I was not likely to continue a correspondence with one who could be guilty of such impertinence as this, and of course took no notice of his Letter. When at Norwich, on my way to the Reception-room of the *British Association*, the morning after my arrival, I fell in with Professor Harley, in conversation with my friend Dr. Ginsburg, when the Professor told me I had not acted fairly with his young friend. This led me to address a Letter to him, in explanation, and subsequently I gave Dr. Ginsburg Mr. Toller's impertient Letter, that he might let the Professor have a perusal of it, and he did peruse it. But Professor Harley never replied to my Letter. Some days afterwards, when on my way home, I accidently fell in with Professor Harley at the Ely Railway Station, when he contrived to stumble out a very lame apology, for taking no notice of my Letter. At the meeting of the *British Association* at Dundee, under the Presidency of His Grace the Duke of Buccleuch, I offered, and was wishful, to read two Papers in Section A, on mean proportionals. The Reverend Professor Harley will remember the important part he played, in preventing me from reading those Papers.

It will depend upon the treatment this communication receives at your hands, whether I do, or do not, trouble you with another But under any circumstances, when I publish, it will be in the form of a series of Letters addressed to you, with your replies so far as they go. In my next Letter I shall prove, that parallelograms of unequal perimeters may contain equal areas, and that parallelograms of equal perimeters may contain unequal areas.

Respectfully yours,

JAMES SMITH.

THE REV. PROFESSOR WHITWORTH.

Now, Sir, Mr. J. M. Wilson, in his Treatise on "*Elementary Geometry*," observes:—"Unsuggestiveness is a great fault in a text-book. Euclid places all his theorems and problems on a level, without giving prominence to the master-theorems, or clearly indicating the master-methods. He has not, nor could he be expected to have, the modern felicity of nomenclature. The very names of *super-position, locus, intersection of loci, projection, comparison of triangles,* do not occur in his treatise. Hence, there already exists a wide gulf between the form in which Euclid is read, and that in which he is generally taught. Unquestionably the best teachers depart largely from his words, and even from his methods. That is, they use the work of Euclid, but they would teach better without it. And this is especially true of the application to problems. Everybody recollects, even if he have not the daily experience, how unavailable for problems a boy's knowledge of Euclid generally is. Yet this is the true test of geometrical knowledge; and problems and original work ought to occupy a much larger share of a boy's time than they do at present."

From this quotation it is perfectly obvious, that Mr. J. M. Wilson not only attaches the *utmost importance* to constructive geometry; but also to "*super-position, locus, intersection of loci, projection*," and "*comparison of triangles*." Now, Sir, I shall proceed to make a "*comparison of triangles*, and such a comparison as *ought to command* your careful attention, as a professional "*recognised Mathematician*," and "*reasoning geometrical investigator*." The triangles are obtained—as a matter of course—by *constructive geometry*, and it is self-evident that they cannot be obtained in any other way. I may fairly infer that you, Sir, in accepting the dignified position of President of the *British Association*, make a profession (if not in words, at any rate by implication) of your desire to advance the cause of scientific truth, so far as it lies in your power, and your scientific knowledge enables you. If I am not *at fault* in drawing this inference, the following facts will receive your careful attention.

First: The coloured triangles F B C, in Diagram IX., and O B D, in Diagram X., are similar to the coloured triangles O B T in Diagrams VII. and VIII., and all these triangles are similar to the triangle O A H, in Diagram I. But, these five triangles are similar to the triangle O B T, in the Diagram in my pamphlet "*Euclid at Fault*," (*Diagram XIV: See Appendix A.*) and in all these triangles the sides that *include* or *contain* the right angle, are in the ratio of 3 to 4, by construction.

Second: The coloured triangle F B E, in Diagram IX., and F E D, in Diagram X., are similar to the coloured triangles H P T, in Diagrams VII. and VIII.;

and all these triangles are similar to the triangle D G H, in Diagram I. But, these five triangles are similar to the triangle H P T, in the Diagram in "*Euclid at Fault*," and in all these triangles the sides that *include* or *contain* the right angle, are in the ratio of 7 to 24, by construction.

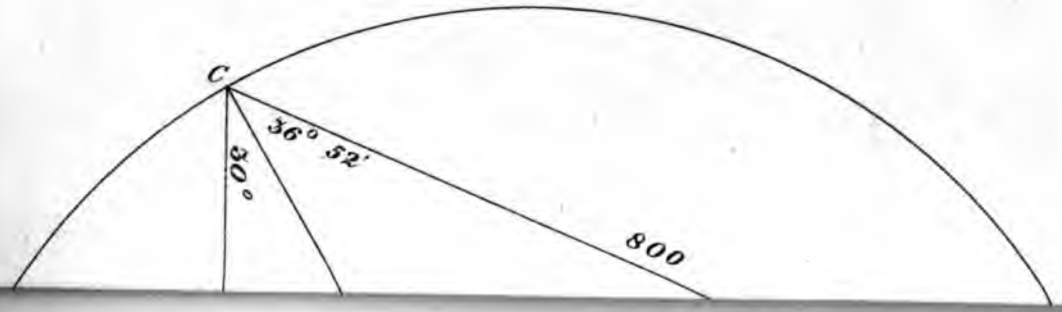
Third: The triangle A F C, in Diagram IX., is a right-angled triangle—Euclid: Prop. 31: Book 3. F B is perpendicular to A C, and the triangles on each side of F B are similar to the whole triangle A F C, and similar to each other—Euclid: Prop. 8: Book 6. The triangle F B A is a right-angled triangle, and the triangle F E A a part of it, is an oblique-angled triangle: and $FE^2 + EA^2 + 2(EA \times EB) = FA^2$—Euclid: Prop. 12: Book 2. We may put any finite value we please on the line A B, which is the generating line of the diagram, make the computations, and demonstrate these facts with arithmetical exactness.

Fourth: It is self-evident that the right angles in the triangles F B C, F B E, and F B A, in Diagram IX., are adjacent angles to the perpendicular F B. In Diagrams VII. and VIII., and also in the Diagram in "*Euclid at Fault*," the triangles O B T are similar to the triangle F B C, in Diagram IX.: and the triangles H P T, in the former diagrams, are similar to the triangle F B E, in Diagram IX.: and it is self-evident that the right angles in the triangles O B T and H P T, in Diagrams VII and VIII., are not adjacent angles, but angles of the parallelograms F B P H. It is self-evident that, in Diagrams VII. and VIII., and also in that in "*Euclid at Fault*," the triangles H P B are right-angled triangles, and the triangles H T B, parts of them, oblique-angled triangles: and by Euclid: Prop. 12: Book 2:

$H T^2 + T B^2 + 2 (T B \times T P) = H B^2$. But, the solution of this theorem is beyond the reach of "*Mathematics, as applied to Geometry by Mathematicians,*" when the diameter of the circles are divided at F, into two parts, in the ratio of 5 to 3. In other words, when $K F = 5$, and $F B = 3$, then, by Euclid : Prop. 8 : Book 6 : and Prop. 35 : Book 3 : $K F \times F B = 5 \times 3 = 15 = H F^2$; therefore, $H F = \sqrt{15}$; and by Euclid : Prop. 47 : Book 1 : $H F^2 + F B^2 = \sqrt{15}^2 + 3^2 = \sqrt{(15+9)} = \sqrt{24} = H B^2$. But, it is beyond the reach of "*Mathematics, as applied to Geometry by Mathematicians,*" to find the arithmetical value of T P, and prove that $H T^2 + T B^2 + 2 (T B \times T P) = \sqrt{24}$. Well, then, with reference to Diagrams VII. and VIII., and also to that in "*Euclid at Fault,*" I have proved, in my correspondence with the Rev. Professor Whitworth, that when $K F = 5$, and $F B = 3$, the arithmetical value of a square on the line H B is not 24, but 24·015625. The following conclusion is irrefragable. All the Propositions of Euclid are not of general and universal application, and Prop. 8 : Book 6 : Prop. 12 : Book 2 : and Prop. 35 : Book 3 : are not the only propositions to which this remark applies ; and this may be readily demonstrated by "*wielding that indispensable instrument of science, Arithmetic.*"

Fifth : In the similar right-angled triangles, to which I have directed your attention in Diagrams I., VII., VIII., IX., X., and that in "*Euclid at Fault,*" the geometrical sines are not all equal. For example : Take Diagrams I. and VIII. When the diameter of the circles Z and X is represented by the number 8, H G the *geometrical* or *natural* sine of the angle H D G in the former diagram = ·84; and T P the *geometrical* or *natural* sine of T H P the

DIAGRAM XI.



similar angle in the latter diagram = ·875. Will you, Sir, venture to tell me that the *trigonometrical* sine of these angles are unequal? Is not the *trigonometrical* sines of these angles = ·28, whatever arithmetical value we may put upon the diameter of the circles? Is not the *trigonometrical* sine of the obtuse angle in the triangles D G H and H P T = ·96?

CONSTRUCTION OF DIAGRAM XI.

Let A and B denote points dotted at random. Join these points, and on A B describe the equilateral triangle O A B. With O as centre and O A or O B as interval, describe the circle; and produce B O to meet and terminate in the circumference of the circle at the point C. Let fall the perpendicular C A, and join O A. With C as centre and C G, = $\frac{4}{5}$ (C B), as interval, describe the arc G D, and join C D, O D, and B D. From the angle D, in the right-angled triangle C D B, let fall the perpendicular D E, and so construct the right-angled triangle D E O.

Now, Sir, it cannot be disputed, that with B as centre and $\frac{3}{5}$ (B C) as interval, we might, from a point in the line C B, describe an arc to meet the circumference of the circle in the direction of D; and if so, an extremity of this arc would meet the circumference at the point D. It is self-evident, that we might construct a diagram representing the trapezium C A B D, and all the straight lines within it, without shewing the circumscribing circle, or the arc D G. But, my good Sir, will you tell me how we could construct the trapezium without the aid of a circle, or two arcs of a circle?

Well, then, the coloured right-angled triangle D E O in Diagram XI. is similar to the triangles D G H and AFM.

in Diagram I.: similar to the triangle E D C in Diagram V.: similar to the triangles H P T in Diagrams VII. and VIII.: similar to the triangle F B E in Diagram IX.: similar to the triangle F E D in Diagram X.: and, similar to the triangle H P T in the diagram in "*Euclid at Fault;*" that is to say, in all these triangles the sides that *include* or *contain* the right angle are in the ratio of 7 to 24, by construction. But further: Referring to Diagram II., it is self-evident that we may join B E, and if B E be joined, then, B O E and D E O, in Diagram XI., will be similar triangles, and it follows, that the triangle B O E is similar to all the others. How could such constructions be possible, if there were no definite relation between the superficies of rectilinear and curvilinear figures? Have I not shewn that circles and squares of the same superficial area may be "*isolated and exhibited*" by means of Diagram V.? Can you, Sir, or any other Mathematician controvert my proofs? Impossible!! It follows then, of necessity, that there is a definite relation between the diameter and circumference in every circle; and that $\frac{1}{3{\cdot}125}$ is the true expression of the ratio between them!!

Now, Sir, referring to Diagram XI., you will observe, that I have put an arithmetical value on every straight line in the diagram: and I have given the value of every rectilinear angle in the figure, expressed in degrees. It is self-evident, that we may halve, quarter, double, or quadruple the arithmetical values of the straight lines, but this could in no way affect the values of the angles. The Rev. Professor Whitworth disputes my values of the angles, and insinuates—for he goes into no proof—that the value of the angle O D E is less than 16°

16′. My correspondent, the Rev. Geo. B. Gibbons, disputed the values of the angles as I have given them, and it was in dealing with his arguments that I was led to adopt this very diagram, simply omitting the circle and the arc G D. The following Letters were written more than two years ago, and will speak for themselves. I am certain Mr. Gibbons will admit that he received such Letters, and may have the originals in his possession :—

BARKELEY HOUSE, SEAFORTH,
4th January, 1867.

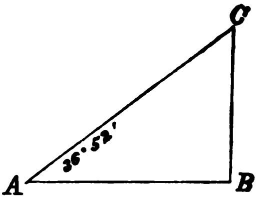
MY DEAR SIR,

Your favour of the 1st instant, which, from some cause, has only just come into my hands, has had my careful attention.

The first paragraph runs thus:—"*I did not expect to give such a shock to the senses, but I must repeat my astounding assertion, viz. In a right-angled triangle, if a side and angle be given (in finite terms) the other sides will not be expressed in finite terms. One side may be, but not both.*" On this point we are of course at issue, and I will proceed to shew you that this "*astounding assertion*" (these words are yours not mine) is an assumption of a thing to be proved, minus the proof. You may *fancy* you have given the proof in the next paragraph of your Letter. We shall see! How many times in the course of our Correspondence have you charged me with, and found fault with me for, assuming the thing to be proved? Why, Sir, so recently as in your Letter of the 24th December last you say:—"*I think it is one of your chief faults, as an investigator of geometrical results, that you employ very complicated figures to perform what* (if it can be done at all) *can be accomplished by easy ones, and the very multiplicity of such drawings leads you off from the fact, that you have never shewn the equality, either in perimeter or area, of a rectilineal and a curvilinear figure. You have assumed it, I grant.*" Now, Sir, in the face of the "*astounding assertion*" contained in this quotation, you are obliged to admit after all, that I have "*shewn correctly the consequences that would follow from* *my assumption*," or I should rather say, from what you are pleased to call my assumption. I reassert what I stated in my last communication. "*In my Letter of the 13th November, I have given you a demonstration of the equality in area, of a rectilineal and curvilinear figure, as plain* and *palpable as that 2 and 2 make 4*."* Your "*astounding assertion*" does not prove the falsehood of mine. To prove my *assertion* I have adopted and applied the simplest principles of geometry, and as a "*Christian and a gentleman*" your duty becomes plain and simple, and that is, to controvert my proof. You surely cannot have worked up your imagination into the belief of the *fancy* that you can *falsify* geometrical truth by *dogmatical assertion.*

I will now proceed to shew you some of "*the consequences that would follow*" from your "*astounding assertion.*"

The second paragraph of your Letter runs thus:—"*Generally, in a right-angled triangle, if the sides are all exact numericals, the angles will not be exactly found, and conversely.*" You then proceed to give what you *fancy* to be a double proof.



First proof: "*Let the sides be 3, 4 and 5, exact. Sin. A =* $\frac{3}{5} = \cdot 6$, *which belongs to no angle exactly assignable.*"

With reference to this proof I may observe:—It is unquestionably true that Sin. A = ·6. But, it is equally true that Cos. A = $\frac{4}{5} = \cdot 8$; and $\text{Sin.}^2 \text{A} + \text{Cos.}^2 \text{A} = \cdot 6^2 + \cdot 8^2 =$ unity. It is at this point your *fancy* steps in. By Hutton's Tables Sin. A is ·5999549 and Cos. A is ·8000338; therefore, $\text{Sin.}^2 \text{A} + \text{Cos.}^2 \text{A} = \cdot 5999549^2 + \cdot 8000338^2 = \cdot 9999999631664 5$, a close approximation to unity; and we may make the approximation as close as we please by extending the number of decimals. Thus, according to Hutton, $\text{Sin.}^2 \text{A} + \text{Cos.}^2 \text{A}$ is not expressible in finite terms, and so forsooth, according to your *logic*, the angle A cannot be an angle of 36° 52'. In the face of these facts can you wonder at the *shock to my senses*, when in your

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* This demonstration was by means of a geometrical figure similar to Diagram V., but omitting the circle Y and its inscribed square. Let Mathematicians prove, that the square R P K E and the circle Z are not equal in superficial area.

Letter of the 24th December you made the "*astounding assertion ?*" "*Antecedently to any such investigation, we must remember that for any given value of A B, the sides C B and A C will not be found exactly.*"

Proof second: "*Let the angle A be exact, say* 36° 52′, *and side* $B\,C = 10$. $\frac{B\,C}{A\,C} = Sin.\ 36° 52′$, *or* $A\,C = 10\,Cosec\ 36° 52′$, *which is not finite.*"

With reference to this proof I may observe:—If the angle A be an angle of 36° 52′, $\frac{\mathrm{A\,C}}{\mathrm{B\,C}}$, that is, $\frac{\text{hypothenuse}}{\text{perpendicular}}, = \frac{5}{3} = 1\frac{2}{3}$, is the cosecant of the angle A, and is *the fractional expression of a finite arithmetical quantity.* Will you dare to dispute *the fact*, because we cannot give decimal expression to this arithmetical quantity in finite terms? I trow not!! Well, then, if $\mathrm{B\,C} = 10$, then, $10\left(\frac{\mathrm{A\,C}}{\mathrm{B\,C}}\right) = \frac{5}{3}\left(10\right) = 16\frac{2}{3} = \mathrm{A\,C}$; and, by analogy or proportion, B C : A C : : B C : 10 cosecant of angle A; that is, 3 : 5 : : 10 : $16\frac{2}{3}$; and since, by hypothesis, the sides of the triangle are 3, 4, and 5, exact, it follows of necessity, that $\mathrm{A\,C} = 16\frac{2}{3}$, neither more nor less. Now, by Hutton's Tables, the cosecant of the angle A, that is, the cosecant of an angle of 36° 52′ is 1·6667920; therefore, 10 times cosecant of angle A = 10 (1·6667920) = 16·667920, and is greater than the indisputable arithmetical value of A C, the hypothenuse of the triangle, which is absurd, and so, your "*astounding assertion upsets itself.*"

I will now proceed to point out what "*I think is one of your chief faults as an investigator of geometrical results.*" You employ the higher branches of Mathematics in the course of your investigations, but in such a way as "*leads you off from the facts*" of simple geometrical truth. To make my meaning perfectly intelligible:—In the problem I gave you for solution in my Letter of the 22nd December, I assumed the angle A to be an angle of 36° 52′, and the side B C = 600 miles. The admissibility of this assumption you do not venture to dispute, nor dare you do so; on the contrary, you admit the premisses, and proceed to make an attempt to solve the problem. Well, then, if A be an angle of 36° 52′, C is an angle of 53° 8′. Now, by Hutton's Tables, the Log-sin. of the angle A is

9·7781186; and, on referring to the Logarithms of numbers, we find that ·7781186 is the nearest Logarithm to the natural number 600, the given value of B C. The Log-sin. of the angle C is 9·9031084, and by the Logarithms of numbers, ·9031084 is the nearest Logarithm to the natural number 800, the value of A B; and, $\sqrt{BC^2 + AB^2} = \sqrt{600^2 + 800^2} = 1000 = AC$; and 800 and 1000 are the true arithmetical values of the sides A B and A C, when A is an angle of 36° 52′, and B C = 600. If you could but see it, my dear Sir, by your mode of reasoning, you put the cart before the horse, and attempt to make the higher branches of Mathematics (Mathematics are based on, and are merely an extension of, the principles of common Arithmetic) over-ride the unerring laws of Arithmetic. You know as well as I do, that in all mathematical investigations we must necessarily appeal to simple Arithmetic for our final results.

The enclosed diagram *(see Diagram XI.)*, which I send you for an especial purpose, is a fac-simile of that in my Letter of the 29th December, with the simple addition of the line O D, and I hardly think you will venture to charge me with troubling you with a complicated geometrical figure.*

The lines O A, O B, O D, and O C are equal. (Of this fact you may readily convince yourself, by describing a circle with O as centre, and any of these lines as radius. You will find that all the outer angles of the figure will touch the circumference of the circle.) These lines sub-divide the trapezium C A B D into four parts, and divide each of the outer angles into two angles. The four angles at the centre of the figure are together equal to the eight outer angles = 360°, and each of the angles at the centre is the apex of an isosceles triangle.

Now, my dear Sir, assuming D B, the sine of the angle D C B, to represent 600 miles, I have given, on the face of the diagram, what I believe to be the *exact* values of every line and angle in the figure. If you are right in your "*astounding assertion*," these values must be erroneous; but if so, *you*, as a Master in Geometrical and Mathematical science, can have no difficulty in making the necessary corrections. May I respectfully beg of you to do so; state them on the face of the diagram, and return it to me. I am sure you cannot have the slightest hesitation in complying with this very reasonable request, if practicable.

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* The reflective geometrical reader will understand why no curved lines were shewn in these diagrams.

Waiting your reply, and wishing you a very happy new year,

Believe me, my dear Sir,

Very truly yours,

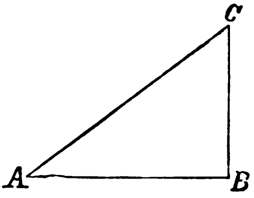
THE REV. GEO. B. GIBBONS, B.A. JAMES SMITH.

BARKELEY HOUSE, SEAFORTH,
9th January, 1867.

MY DEAR SIR,

I wrote you a Letter dated the 4th, in reply to yours of the 1st instant, but for certain reasons did not post it till the 5th, so that it should not come into your hands before Monday morning. In that Letter I have, as I think, exposed the fallacy of your "*astounding assertion*," viz. "*Generally, in a right-angled triangle:—If the sides are all exact numericals, the angles will not be exactly found, and conversely.*" I shall be greatly surprised if you find it *practicable* to controvert the proofs I have given you of the absurdity of such an assertion. To do so you will have to demonstrate that the higher branches of Mathematics over-ride, not only the principles of common Arithmetic, but the plainest and simplest truths of Geometry. Now, Sir, while I maintain that if the sides of a right-angled triangle "*are all exact numericals*" the angles may be exactly found, I also maintain that the higher branches of Mathematics rightly applied, so far from contradicting, on the contrary, confirm and establish the truths of Geometry. One example will, I think, suffice to convince you of this fact:—

Let the sides of a right-angled triangle C B A *be* 3, 4, *and* 5 *exact*, and let the side A B be represented by 800 miles. Then, $\frac{AB}{AC} = \frac{4}{5} = \cdot8$ is the Sin. of the angle C, and C is an angle of 53° 8′. The Log-sin. of the angle C is 9·9031084, by Hutton's Tables.



Now, referring to the Logarithms of numbers, we find that

·9031084 is the nearest Logarithm to the natural number 800. The Logarithm of the natural number 800 is 9·9030900, and these two Logarithms agree within 1 at the fifth place of decimals. Well, then, I put the question to you:—Is it conceivable, my dear Sir, that Logarithms which are admitted to be mere approximations, could be expected to approximate *more* closely ?

I must now carry you back to your Letter of the 19th October last, in which you make some other "*astounding assertions.*" In that Letter you say:—"*There is no angle expressible in finite terms whose Sin. is* ·28 *or* ·6. *Sin.* 16° 15′ *is rather less, and Sin.* 16° 16′ *is rather greater than* ·28; *and Sin.* 36° 52′ *is rather less, and Sin.* 36° 53′ *rather greater than* ·6." Now, my dear Sir, it appears to me the simplest thing imaginable, to expose the fallacy of such assertions, and this I shall proceed to do.

The enclosed diagram (*See Diagram XI.*) is a fac-simile of that contained in my Letter of the 4th instant, with the simple addition of the straight line D E, drawn from the point D, perpendicular to the line C B; and C B is the hypothenuse of, and common to, the triangles C A B and C D B. Euclid : Prop. 31 : Book 3 : will enable you to convince yourself, that the angles C A B and C D B are right angles, if with O as centre and O A as radius you describe a circle.*

Now, C B : B D in the ratio of 5 to 3; and, C B : C D in the ratio of 5 to 4, by construction; therefore, the sides C D and D B, which contain a right angle, are in the ratio of 4 to 3, by construction. Thus, the sides of the triangle are represented by "*exact numericals;*" and assuming C D = 800 miles, C D : D B : : 4 : 3, which makes D B = 600 miles; and C D : C B : : 4 : 5, which makes C B = 1000 miles.

Again : C D : D E : : D B : B E; and E C : E D : : E D : E B; therefore, C B : C D : : C D : C E; therefore, the triangles C E D and D E B are equiangular, and consequently have their sides about the equal angles proportional, and are therefore similar. Thus, by Euclid : Prop. 8 : Book 6 : the triangles C E D and D E B are similar to the triangle C D B and to each other, and it follows of necessity, that C E : E B : : $4^2 : 3^2$, that is, 640 : 360 : : 16 : 9.

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* The reflective reader will now see the writer's reason for omitting the circle from the diagram, in his communications with the Rev. Geo. B. Gibbons.

Again : O D B is an isosceles triangle of which O D and O B are the equal sides, and the triangle is divided into two right-angled triangles by the line D E. Now, D B : D E in the ratio of 5 to 4, and 5 : 4 : : 600 : 480; therefore, $DE = 480$ miles, when $DB = 600$ miles. But, D E : E B in the ratio of 4 to 3 ; therefore, 4 : 3 : : 480 : 360 ; therefore, $EB = 360$ miles, when $DB = 600$ miles. Well, then, twice the product of 4 and $3 = 2\,(4 \times 3) = 24$, and $4 + 3 = 7$. From these facts we obtain the ratios that exist between the sides of the right-angled triangle D E O. Thus : D E : E O in the ratio of 24 to 7, and E O : D O in the ratio of 7 to 25 ; therefore, D E : D O in the ratio of 24 to 25. But, 24 : 7 : : 480 : 140 ; therefore, $EO = 140$ miles ; and 24 : 25 : : 480 : 500 ; therefore, $DO = 500$ miles. Proof : $\sqrt{DE^2 + EO^2} = \sqrt{480^2 + 140^2} = \sqrt{230400 + 19600} = \sqrt{250000} = 500$. This is in harmony with the theory of commensurable right-angled triangles, to which I directed your attention in my Letter of the 8th August last, and to which I would now call your especial attention. It is possible you may have destroyed or mislaid that Letter ; if so, I shall be glad to let you have a copy of it. But even this is not requisite. It is only necessary you should give my Letter of the 13th November last, the attention it ought to command from you as an honest and earnest enquirer after geometrical truth, to convince yourself of the absurdity of, at any rate, one of your "*astounding assertions.*"

Again : $\frac{EB}{DB} = \frac{3}{5} = \frac{360}{600} = 6 =$ Sin. of angle E D B, and $\frac{DE}{DB} = \frac{4}{5} = \frac{480}{600} = {\cdot}8 =$ Cos. of angle E D B. $\frac{EO}{DO} = \frac{7}{25} = \frac{140}{500} = {\cdot}28 =$ Sin. of angle O D E, and $\frac{DE}{DO} = \frac{24}{25} = \frac{480}{500} = {\cdot}96 =$ Cos. of angle O D E. But, we may obtain the Sines and Cosines of the angle in another way, and I must here *reiterate* what I have proved in my Letter of the 13th November. $\frac{DB}{CB} = \frac{3}{5} = \frac{600}{1000} = {\cdot}6 =$ Sin. of angle B C D, and $\frac{CD}{CB} = \frac{4}{5} = \frac{800}{1000} = {\cdot}8 =$ Cos. of angle B C D, and, $(\text{Sin.}^2\ BCD) + (\text{Cos.}^2\ BCD) =$ unity. But, $\frac{(\text{Sin. } BCD) + (\text{Cos. } BCD)}{5} = \frac{{\cdot}6 + {\cdot}8}{5} = \frac{1{\cdot}4}{5} = {\cdot}28 = \frac{EO}{DO} =$ Sin. of angle E D O, and $\{2\,(\text{Sin. } BCD) \times (\text{Cos. } BCD)\}$

$= 2(\cdot 6 \times \cdot 8) = (2 \times \cdot 48) = \cdot 96 = \frac{DE}{DO} =$ Cos. of angle E D O, and (Sin.^{2} E D O) + (Cos.^{2} E D O) $= \cdot 28^2 + \cdot 96^2 = \cdot 0784 + \cdot 9216$ = unity.

How could these things be, if your "*astounding assertions* were based on geometrical truth? I will ask you further:—In the face of these facts, which I cannot but think will be as plain and palpable to you as that 2 and 2 make 4, how can you hesitate to admit your error, and withdraw your absurd assertions, "*without offence to your own conscience?*"

I have divided the right angle, in the triangle C D B, into three distinct angles. The angle C D O = 36° 52′. The angle O D E = 16° 16′. The angle E D B = 36° 52′. The three angles are together equal to the right angle C D B = 90°. That these angles have a value is indisputable, and, as I think, their Sines are represented by the arithmetical symbols ·6 and ·28 exactly. If you still persist in asserting that ·6 and ·28 "*belong to no angles exactly assignable*," it is at least your bounden duty, as a candid controversialist, to demonstrate the fallacy of my conclusions, and furnish the proofs of your own.

In conclusion:—It occurs to me, that I cannot do better, by way of bringing our controversy *on the solution of a right-angled triangle*, to an issue, than give you the following problem for solution:—*Bisect the angle* O D E, *and find the arithmetical value of the Sine of half the angle.*

Believe me, my dear Sir,
Yours very truly,
JAMES SMITH.

THE REV. GEO. B. GIBBONS, B.A.

BARKELEY HOUSE, SEAFORTH,
14th January, 1867.

MY DEAR SIR,

Your Letter of the 10th, in reply to mine of the 4th and 9th inst., came to hand on Saturday evening. I have given it my careful attention, and can assure you that, the more I have pondered over it the greater has been my surprise. The Pope of Rome could not have excommunicated a heretic from the pale of the church with greater "*sang-froid*," than you have presumed to excommunicate me from the *charmed* circle of Mathematical "*authority*." You charge me with childishness, with ignorance, nay more, by implication you charge me with DISHONESTY; for, you *rashly* assert that I "*impugn*" the "*Tables of Sines*" which, as you say, "*I have never investigated.*" Had I left myself open to the imputation of such a charge, it would indeed be a question, whether knavery or folly was the most prominent feature in my character and conduct. Fortunately, however, your "*astounding assertions*" *upset themselves*, and I tell you, Sir, without any hesitation, that you are rapidly drifting on to the shoals of Mathematical dishonesty, and that if you do not speedily alter your course, you will most assuredly make shipwreck of your reputation, both as a Geometer and a Mathematician. I do not make these statements *rashly*, and as an honest man I am *bound* to warn you of the precipice upon which you are standing, by furnishing the necessary evidence of the facts, and this I shall proceed to do.

I must now refer you to the diagram enclosed in my Letter of the 9th inst. *(See Diagram XI.)*

In the triangle C D B the sides are in the proportion of "3, 4, *and* 5 *exact*," by construction; and I have proved in my last Letter that the triangles C D B, C D E, and D E B are similar triangles; therefore, the sides of the triangle D E B are *in the proportion of* 3, 4, *and* 5 *exact*. These facts you have not attempted to dispute, nor dare you do so, for you know as well as I do, that were you to make any such attempt, it could only result in *convicting yourself* of gross ignorance, with reference to some of the simplest truths of geometry.

Well, then, you say: "*In your diagram take the triangle B E D. If B is given* 53° 8′ *and B D is given* 600. *Then*:

$$\begin{array}{ll} D\,E = 600 \text{ Sin } 53^\circ\, 8' & \cdot 8000338 \\ \quad = 480 \cdot 02028 & \underline{\qquad\quad 600} \\ & 480 \cdot 0202800 \end{array}$$

and not 480 *exactly, as you put it.*"

Now, Sir, because the sides of the triangle C D B are in the proportion of 3, 4, *and* 5, *exactly*, by construction, and because the triangles C D B and B E D are similar triangles, it follows of necessity, that the sides of the triangle B E D are in the proportion of 3, 4, *and* 5 *exactly;* therefore, $\frac{\text{D E}}{\text{D B}} = \frac{4}{5} = \cdot 8$ = Sin of angle D B E, and 600 (Sin D B E) = 600 × ·8 = 480 = D E. Thus, 5 : 4 : : D B : D E, that is, 5 : 4 : : 600 : 480. This you assume to be nonsense, and would have me believe that 5 : 4 : : 600 : 480·02028 ; and I put the following plain question to you: Do you think you can find a first class school-boy who could be guilty of folly and absurdity equal to this ?

Again: You say: "*In the triangle D O E. If O D E is given* 16° 16′ *and O D* = 500. *Then:*

O E = 500 Sin 16° 16′ ·2801083
= 140·05415 500

140·0541500

not 140 *exactly, as you have it.*"

Now, Sir, because the sides of the triangle B E D are in the proportion of 3, 4, and 5 exactly, it follows of necessity, that the sides of the triangle D E O are in the proportion of 24, 7, and 25 exactly. These *facts* may or may not be within your comprehension, but whether they are, or are not, the *facts* are incontrovertible. Now, $\frac{\text{E O}}{\text{D O}} = \frac{7}{25} = \cdot 28$ = Sin of angle O D E, and $\frac{\text{D E}}{\text{D O}} = \frac{24}{25} = \cdot 96$ = Cos. of angle O D E ; and $\text{Sin}^2 + \text{Cos}^2 = \cdot 28^2 + \cdot 96^2$ = unity. But, 500 (Sin O D E) = 500 × ·28 = 140 = E O, and 500 (Cos O D E) = 500 × ·96 = 480 = D F. ; therefore, $\sqrt{\text{E O}^2 + \text{D E}^2} = \sqrt{140^2 + 480^2}$ = 500 = D O. Thus, D E : E O : : 24 : 7, that is 480 : 140 : : 24 : 7 ; and D E : D O : : 24 : 25, that is, 480 : 500 : : 24 : 25.

Now, for the sake of argument I shall assume your conclusions, and mark the result ! Then : $\frac{\text{E O}}{\text{DO}} = \frac{140 \cdot 05415}{500} = \cdot 2801083$ = Sin of angle O D E ; and $\frac{\text{D E}}{\text{D O}} = \frac{480 \cdot 02028}{500} = \cdot 96004056$ = Cos of angle O D E; therefore, (Sin² O D E) + (Cos² O D E) = $\cdot 2801083^2 + \cdot 96004056^2 = \cdot 0921677876845 1136 + \cdot 07846065972889 = 1 \cdot 0001385362740036$, an arithmetical quantity greater than unity.

This would sap, undermine, and shiver to atoms the very foundation of Trigonometry ; and so, your "*astounding assertions upset themselves.*"

Well, then, to establish the charges you have brought against me of childishness, ignorance, and (by implication) dishonesty, you must prove, and it becomes your bounden duty to prove—

First: That it is not true of all right-angled triangles that Sin.^{2} + Cos. = unity.

Second: That the triangle D E O is *not* a right-angled triangle, and that the sides are *not* in the proportion of 24, 7, and 25 *exactly.*

Third: That the arithmetical value of the Sin. and Cos. of the angle O D E *cannot be* ·28 and ·96, and that their true and exact values *must be* ·2801083 and ·9599684, as given in Hutton's Tables.

But, Sir, *assuming* (if I can be forgiven for *assuming* an impossibility) your mathematical capacity to get rid of these difficulties, you would still have something more to do. You would have to demonstrate the absurdity of the THEORY I have propounded with reference to commensurable right-angled triangles, as to which I directed your especial attention in my Letters of the 8th August and 13th November last. Will you dare to tell me that this THEORY is not of any importance in the solution of a right-angled triangle? Now, Sir, I must remind you that this THEORY you have so far treated with the most profound contempt, and I assert, without hesitation, that you have adopted a similar course throughout our correspondence, with every argument that you have found it inconvenient to grapple with.

On the third page of your Letter you observe:—"*At any rate I shall be silent if you like to persist in saying, in the face of multiplied calculations, that Sin.* 53° 8′ = ·8 *exactly, or the like of other arcs you name.*" If you choose to be silent, I can't help it, and the threat "*troubles me not;*" but, I may tell you that I cannot conceive how you can decline to reply to this communication, "*without offence to your own conscience.*"

In conclusion: I beg to repeat the request made at the close of my last Letter. "*Bisect the angle O D E, and find the arithmetical value of the sine of half the angle.*" I can assure you there is more to come out of this than was ever "*dreamed of*" in your mathe-

matical Philosophy, and this I shall be prepared to prove when the proper time arrives. Waiting your reply,

Believe me, my dear Sir,

Very truly yours,

JAMES SMITH.

THE REV. GEO. B. GIBBONS, B.A.

It is self-evident, that C B is the hypothenuse of, and common to, the triangles C A B and C D B. Now, if Euclid be right in the Theorem: Prop. 31: Book 3: C A B and C D B are right-angled triangles: and, if Euclid be right in the Theorem: Prop. 47: Book 1: $(CA^2 + AB^2) = (CD^2 + DB^2)$; and this equation or identity $= CB^2 =$ area of a circumscribing square to the circle. I maintain that these two Theorems of Euclid are as irrefragable, as the Theorem: Prop. 32: Book 1. Now, O D, O C, and O B, are equal, for they are radii of the circle, and it follows that $4\ (OD^2) = CB^2$. But, you will observe that Mr. Gibbons makes D E greater than 480, and E O greater than 140, when C B the diameter of the circle = 1000. If his data and reasoning be sound, O D a radius of the circle, must be greater than 500, when the diameter of the circle = 1000. This would make $4\ (OD^2)$ greater than the area of a circumscribing square to the circle. Can you fail to perceive, that this would make the Theorems of Euclid: Prop. 47: Book 1: and Prop. 31: Book 3: inconsistent with each other? This is *only* one of the *absurdities* into which *a Mathematician may be led by the misapplication of Mathematics to pure Geometry.*

It is self-evident, that the line AB is equal to half the side of an inscribed regular hexagon to a circle of which C B is the radius; and I have given the values of all the sides of the triangles in Diagram XI., when C B the diameter = 1000. Now, the two terms of a ratio may be divided by the same number, without altering the ratio itself; and it follows, that we may divide all the sides of the triangles by 500 without altering the ratios of side to side: and, if we so divide them, then C D = 1·6: DE = ·96 : D B = 1·2 : C B = 2 : O E = ·28 : and E B = ·72. Hence: $(DE^2 + EO^2) = (·96^2 + ·28^2) = (·9216 + ·0784) = 1 = OD^2$; therefore, $\sqrt{1} = 1 =$ OD, OA, OC, and OB: and, $CD^2 + DB^2 = (1·6^2 + 1·2^2) = (2·56 + 1·44) = 4 = CB^2$; therefore, $\sqrt{4} = 2 = 2$ (O D) = C B. The ratios of side to side in the triangles are unaltered; that is to say, D B : C D : : 3 : 4 : and, O E : E D : : 7 : 24. Now, *assuming* Mr. Gibbons' data and reasoning to be sound, it follows, that no definite relation exists or can exist between the sides of the triangles, C E D, D E B, and CDB: and between the sides of the triangle D E O; and it would follow, that no definite relation could possibly exist between the diameter and circumference in a circle.

Will you, Sir, or any other "*recognised Mathematician*," venture to tell me that I have not proved a definite relation to exist between the sides of the triahgles? Will any Mathematician dare to tell me that I have not demonstrated these ratios? I trow not! Well, then, it follows of necessity, although "*recognised Mathematicians*" either *can* not, or *will* not see it, that a finite and determinate relation must exist between the diameter and circumference in every circle.

I have never said that the Theorems of Euclid:

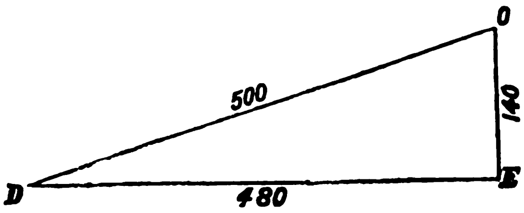
Prop. 12: Book 2: and, Prop. 8: Book 6: are not true under *any circumstances.* What I have said *is*, that they are not of "*general and universal application*," and therefore are *not true* under *all circumstances.* I *know*, that with reference to Diagram XI.: $\{DO^2 + OC^2 + 2(OC \times OE)\} = CD^2$: and I also *know* that the triangles on each side of D E are similar to the whole triangle C D B, and to each other; and I *know* that we can work out these results with arithmetical exactness, whatever value we may put upon the radius of the circle: it matters not whether it be a finite quantity, or be represented by the square root of a finite quantity. But, assuming Logarithmic Tables of Sines, Cosines, &c. to be infallible, which Mr. Gibbons does; of course, by computations based upon this fallacious assumption, he makes D E greater than 480, and O E greater than 140, when C B the diameter of the circle = 1000. Let Mr. Gibbons prove his values of the lines D E and O E, by demonstrating that $\{DO^2 + OC^2 + 2(OC \times OE)\} = CD^2$: and that the triangles on each side of D E are similar to the whole triangle C D B and to each other. He will find that he "*upsets*" the Theorems of Euclid: Prop. 12: Book 2: and, Prop. 8: Book 6. Such are the absurdities and inconsistencies into which a Mathematician may be led, *by a misapplication of Mathematics to pure Geometry.*

You must not imagine that the Rev. Geo. B. Gibbons is not a "*recognised Mathematician.*" He is the intimate friend of your acquaintance, Professor Adams, the well known Astronomer, and the Professor *knows* of my long correspondence with that gentleman, and can satisfy you on this point. When attending the last meeting of the British Association at Norwich, I had the opportunity of a short conversation with Professor Adams, and told him that before the next meeting of the Association, I should bring out a work and demonstrate the true ratio of diameter to circumference in a circle, by means of angles.

Now, Sir, unless you and other "*recognised Mathematicians*" can prove the values I have given of the angles, in the geometrical figure represented by Diagram XI. to be false, which I *know* you can't: as *gentlemen*, and *men of honour*, "*you should not be ashamed to admit*" that I have furnished the proof.

I wrote Mr. Gibbons several Letters between the 14th and 28th January, 1867. These, however, I must pass over: but, I quote the following from a Letter of the latter date:—

> "In my Letter of the 9th instant, I brought under your notice a geometrical figure, in which I have introduced two right-angled triangles, having one side D E common to both, (*see Diagram XI.*) and (whether it is, or is not, within your geometrical capacity to comprehend it) I have demonstrated beyond the possibility of dispute or cavil in my Letter of the 14th, that the sides of the triangle D E B are in the ratio or proportion of 3, 4, and 5 exactly; and the sides of the triangle D E O in the ratio or proportion of 7, 24, and 25 exactly. Both letters conclude by my making '*what I am sure you would confess*' to be, a very reasonable request, namely, *Bisect the angle O D E, and find the arithmetical value of the sine of half the angle.*"
>
> "With reference to this request you now say: '*You ask me to bisect the angle O D E, having given the three sides. This is easily done.*'"



"You then proceed to give the following solution of the theorem":—

"$2 \text{ Cos. } \phi = \sqrt{1 + \text{Sin. } 2\phi} + \sqrt{1 - \text{Sin. } 2\phi}$. Call ODE $= 2\phi$, its Sine is $\frac{140}{500} = {\cdot}28$,* and its Cosine, which is Sine of O $= {\cdot}96$.

Hence:

$$2 \text{ Cos. } \phi = \sqrt{1{\cdot}28} + \sqrt{0{\cdot}72} = 1{\cdot}1313708 + {\cdot}8485281 = 1{\cdot}9798989.$$

$$\therefore \text{ Cos. } \phi = \frac{1{\cdot}9798989}{2} = {\cdot}9899494."$$

"Now, Sir, I frankly admit (and that without any mental reservation) that ·9899494 is the arithmetical value of the cosine of half the angle O D E as nearly as it can be given to 7 places of decimals, and without making the equation $\text{Sin.}^2 + \text{Cos.}^2$ greater than unity: and so, we have arrived "*at an agreement*" on this particular point."

"From your premisses you reason to the following conclusion":—

$$\begin{array}{rl} \text{"Cos. } \phi = & {\cdot}9899494 \\ \text{Cos. } 8^\circ\ 8' = & {\cdot}9989415 \\ \hline & {\cdot}0000079 \text{ difference.} \end{array}$$

Which for difference 411 correspondents to about 11,″ so that half the angle O D E = 8° 7′ 49″ nearly."

Now, $2\,(8^\circ\ 7'\ 49'') = 16^\circ\ 15'\ 38''$ nearly, *is*, according to Mr. Gibbons, the value of the angle O D E expressed in degrees.

In a subsequent Letter, Mr. Gibbons computed the value of the angle E D B in the triangle D E B, in a similar way, arriving at the following conclusion:—

$$\begin{array}{rl} \text{"Sin } 2\,\phi = & {\cdot}6000000 \text{ by hypothesis.} \\ \text{Sin } 36^\circ\ 52' = & {\cdot}5999549 \\ \hline & {\cdot}0000451 \text{ for difference } 2326. \end{array}$$

$$\text{So that } 2\,\phi = 36^\circ\ 52'\ \tfrac{451}{2326} \text{ nearly."}$$

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*In Mr. Gibbons' Letter, this was put $\frac{480}{500} = {\cdot}28$. This was obviously a *lapsus*, and I have corrected it; and I am sure Mr. Gibbons will acquit me of ever attempting to catch at, and play with, a *mere lapsus*, in the course of our very long correspondence, With reference to these conclusions, I put the following question to Mr. Gibbons:—"*Is it conceivable that you could have advanced an argument better calculated to prove your assumption of the infallibility of our Mathematical Tables?*" At this time, the question at issue between us was, whether the values of sines, cosines, tangents, Log-sines, Log-cosines, &c., as given in Logarithmic Tables, are or are not correct; I maintain that they are not correct.

Now, Sir, the angles C and D, in the triangle C O D, are equal, for they are angles at the base of an isosceles triangle: and the angle O, at the apex of the isosceles triangle C O D, is greater than a right angle by the value of the angle O D E in the triangle D E O, whatever be that value. But, the angles C and D, in the triangle C O D, are equal to the triangle E D B in the triangle D E B: and, since the triangles on each side of D E are similar to the whole triangle C D B and to each other, it follows of necessity, that the angles C D O and O D E are together equal to the angle B in the triangle D E B.

Well, then, according to Mr. Gibbons, the angle C D O $= 36^\circ\ 52'\ \frac{451}{2356}$ nearly, and the angle O D E $=$ $16^\circ\ 15'\ 38''$ nearly. What is the sum of these two angles? In other words, what is the value of the angle D, in the triangle C D E? Is it not beyond the reach of that "*indispensible instrument of science, Arithmetic,*" to add together these two quantities, and give an assignable and definite value to the angle D, in the triangle C D E? It will be—or at any rate, ought to be—as plain and palpable to you, Sir, as that $2 + 2 = 4$, that Mr. Gibbons' data and reasoning would "*upset*" the Theorem of Euclid: Prop. 32: Book 1: or in other words, would make the three angles of a plane triangle *not* equal to two right angles. This is another of the absurdities into which a Mathematician may be led, *by a mis-application of Mathematics to pure Geometry.*

Now, Sir, when CB, the diameter of the circle, = 1000, the area of a square on C A = 750000; therefore, $\sqrt{750000}$ = 866·025403784 = area of a square on C A, the perpendicular of the right angle C A B, *approximately.* But, $866{\cdot}025403784^2$ is not equal to 750000, and by no extension of the decimals could we ever get to the true area of a square on the line C A, by this process. But, ·5 is the *trigonometrical* sine of the angle C, and *trigonometrical* cosine of the angle B, in the triangle C A B: and, ·8860524, that is, $\sqrt{\cdot 75}$ is the *trigonometrical* sine of the angle B, and *trigonometrical* cosine of the angle C, in the triangle C A B: and the angles C and B are among the very few angles in which the *natural* or *geometrical* sine, and the *trigonometrical* sine, are the same. Hence, the sines and cosines of these angles are correctly given in our Logarithmic Tables of Sines, Cosines, &c., to 7 places of decimals, although the triangle C A B is an incommensurable right-angled triangle.

Now, Sir, we live in an age when the supposed *impossibility* of one day, may become the admitted *possibility* of the next; and of this there have been many remarkable examples in my time. If, half a century ago, a man had foretold that the time would come when we should be able to travel comfortably at the rate of 40 miles an hour, would he not have been thought mad? If any modern discoverer in electricity had, a very short time ago, suggested the *possibility* of an Atlantic Cable conveying messages between this country and America, would he not have been thought mad? and yet, the day has come when Marine cables are conveying messages daily to nearly all parts of the world. Why then should it be assumed, by the existing race of professional Mathematicians, that they have arrived at the *ne plus ultra* of geometrical truth? Well, then, if you decline to give your consideration to the geometrical truths I have brought under your notice, *the day will come*, when other Mathematicians *will;* and when these truths will be universally admitted, and be known to, and comprehended by, every first-class schoolboy. That day may be nearer at hand than I anticipate, but it may be in your time, if not in mine; and then, what will you and such of your compeers of the *British Association* as may be living, think of yourselves?

JAMES SMITH *to* THE REV. PROFESSOR WHITWORTH.

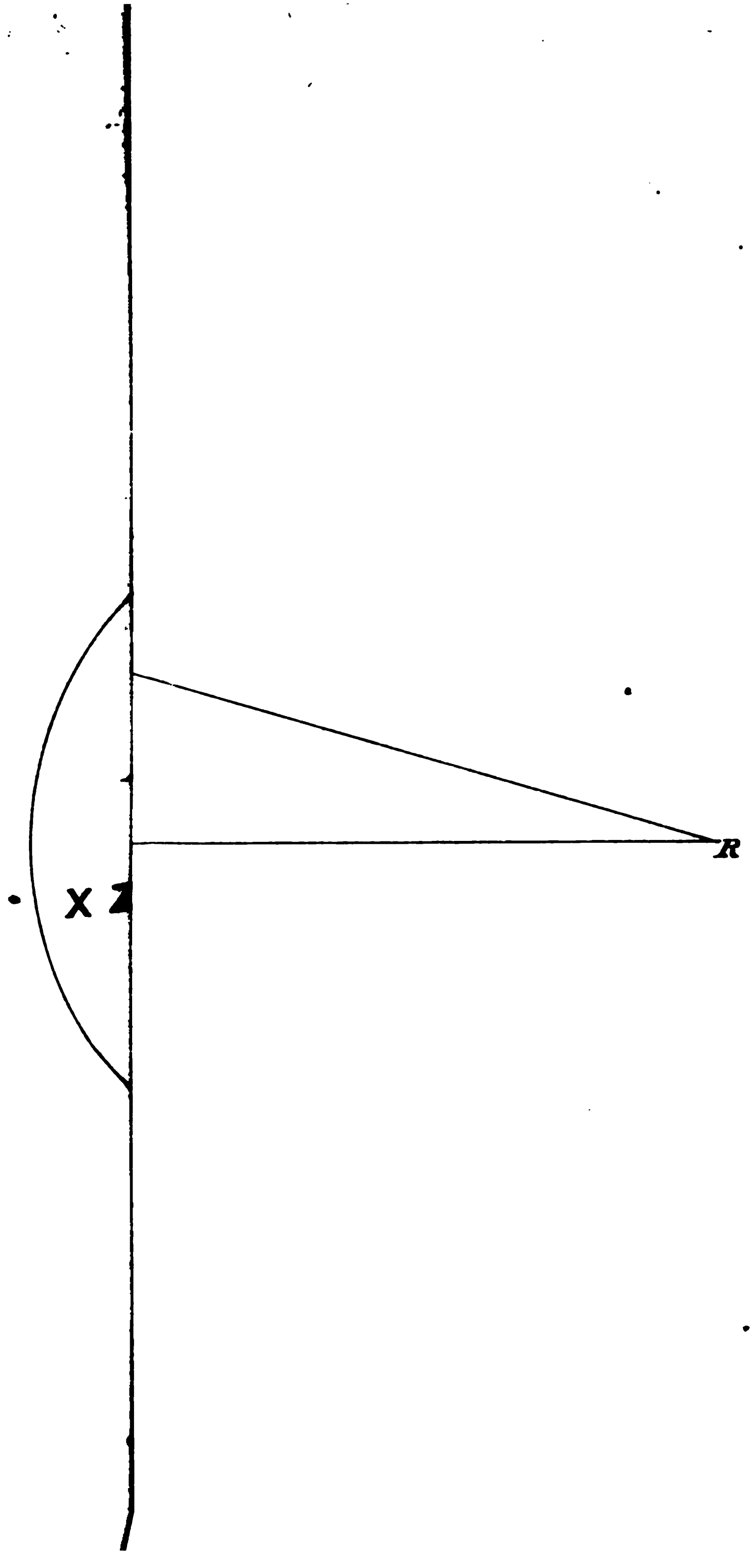
BARKELEY HOUSE, SEAFORTH,
2nd February, 1869.
Posted 8th February.

SIR,

I posted a long Letter, addressed to your private residence, yesterday morning, which I presume would come into your hands when you returned home, on the close of your day's labour at Queen's College. Should you return it unopened in due course—and, to be consistent, you should do so—I shall get it this afternoon.

The following may be taken as the construction of the geometrical figure represented by the enclosed diagram. *(See Diagram XII.)*

With A as centre, and any interval, describe the circle X, and draw the radii A B and A C at right angles to each other. Produce A B to D, making A D equal to three times A B. Produce A C to E, making A E equal to 4 times A C, and join E D. It is obvious that E A D is a right-angled triangle, of which the sides that contain the right angle are in the ratio of 3 to 4, by construction. With A as centre and A D as interval, describe the circle Y. With A as centre and and A E as interval, describe the circle Z. With E as centre and E A as interval, describe the circle X Y. Produce A D to H, making A H equal to twice A D, or 6 times A B. With A as centre and A H as interval, describe the circle X Z. Bisect A E at O, and with O as centre and O A or O E as interval, describe the circle Y Z. The circumference of the circle Y Z cuts the line E D, the hypothenuse of the right-angled triangle E A D, at the point *a*. From the point *a* draw a straight line through the point O, the centre of the circle Y Z, to meet the circumference of the circle at the point *b*, and join *a* A, *b* A, *b* E, and *a* E. It is self-evident that the rectangle E *b* A *a* is divided by the diagonal E A, as well as by the diagonal *b a*, into two similar and equal right-angled triangles. Produce E A, a diameter of the circle Y Z both ways, to meet the circumference of the circles X Y and Z, at the points F and G. From A, the centre of the circle X, draw a straight line through the point *a*, to meet and terminate in the circumference of the circle X Y at the point K, and join F K. From the point K draw a straight line through the point E the centre of the circle X Y, to meet and terminate in the circumference of the circle at the point L, and join L F and L A. Produce E *b* to meet and terminate in the circumference of the circle Z at the point N. Produce E D to meet and terminate in the circumference of the circle Z at the point M. From the point M draw a straight line through the point A, the centre of the circle Z, to meet and terminate in the circumference of the circle at the point N. Join N L, N G, M K and M G. Produce M E, N E, L A and K A to meet and terminate in sides of the rectangles F L A K and E N G M, at the points *c*, *d*, *g*, and *f*. From F A cut off a part P A, making P A equal to E A + A D, or 7 times A B, and join P B. On P B describe the square P B V T. Produce A H to R, making H R equal to 3 times A H, or A R equal to 24 times A B, and join P R. From the point N draw the straight line N *m*


R
X
Z

perpendicular to N M, and from the point M draw a straight line perpendicular to N M to meet L K produced at the point *n*. From the points K and M draw straight lines, K*p* and M *t*, parallel to the line A R, and equal to K L and M N, and join *p t*.

Now, I must premise, that I have not said in the pamphlet "*Euclid at Fault*," or anywhere else, that the Theorems of Euclid: Prop. 12: Book 2: and Prop. 8: Book 6: hold good under NO circumstances. What I have said is, that they are "*not of general and universal application*," and are, therefore, "*not true under* ALL *circumstances*," and this I have proved. In the enclosed diagram we have examples which *hold good* with regard to both these theorems. But, the remarkable geometrical figure represented by the diagram, not only contains within itself the means of demonstrating the value of π, or the true circumference of a circle of diameter unity, but it does more. It enables us to demonstrate the true ratio of diameter to circumference in every circle, by means of angles.

This geometrical figure is a perfect study for Geometers, and it would be absurd to suppose that I could exhaust all its properties within the limits of a letter of reasonable length. I shall, therefore, content myself with pointing out a few of its peculiar properties, and leave it to future Geometers to pursue the enquiry. Let them go to work in any way they please, I defy them to find anything inconsistent with the many proofs I have already given *you*, of the true ratio of diameter to circumference in a circle, by practical or constructive Geometry.

Now, the right-angled triangles N *m* L, M *n* K, and P A R are similar right-angled triangles, and all similar to the right-angled triangle H P T in the diagram in "*Euclid at Fault*;" and are also similar to the right-angled triangles D G H, E D C, F B E, F E D, and D E O, in Diagrams I., V., IX., X., and XI.; that is to say, in all these triangles the sides that contain the right angle are in the ratio of 7 to 24. The triangle E A D is similar to the triangle O B T, in "*Euclid at Fault*," and also similar to the triangles O A H, A B C, F B C, O B D, and C D B, in Diagrams I., V., IX., X., and XI.; that is to say, the sides that contain the right angle in all these trngles, are in the ratio of 3 to 4. But, in the enclosed diagram *(Diagram XII.)*, P A is equal to the sum of E A and A D, and A B is equal to the difference of E A and A D, by construction ; and it follows of necessity, that the sum of the squares of the three sides of the right-angled triangle E A D is equal to the square on P B. Hence : $(P A^2 + A B^2)$ = area of the square on P B, and is equal to the area of the circles X Y and Z : and $(P A^2 — A B^2)$ = area of an inscribed regular dodecagon to the circles X Y and Z. These facts bring into play the binomial theory. But, E A is the radius of, and common to, the circles X Y and Z, and $3\frac{1}{8}$ times the area of a square on E A, is equal to the sum of the squares of the three sides of the right-angled triangle E A D, or the area of a square on P B ; and it follows, that the square P B V T and the circles X Y and Z, are exactly equal in superficial area.

Now, E L, E K, A N and A M are equal, for they are radii of the circles X Y and Z. But, E A is the radius of, and common to, the circles X Y and Z, and it follows of necessity, that the parallelograms L N A E and E A M K are similar and equal parallelograms. Again: The line N M, a diameter of the circle Z, is a side of, and common to, the parallelograms L N M K and *m* N M *n*. But, it is self-evident that *m* N and *n* M are shorter lines than L N and K M, for the latter are the hypotheneuses of right-angled triangles, of which the former are the perpendiculars ; and it follows of necessity, that L N M K and *m* N M *n*, are dissimilar paralleograms. Again : The line K M is a side of, and common to, the paralleograms K M N L and K M *t p*, and K *p* and M *t* are equal to K L and M N, by construction. But, because K *p* and M *t* are not parallel to K L and M N, K M N L and K M *t p*, are dissimilar parallelograms, although their perimeters are equal, by construction. Again : Because F K is parallel to E M, and F L parallel to E N, and the parallelogram E *b* A *a* common to the parallelograms F L A K and E N G M, it follows of necessity, that the three parallelograms are similar, and F L A K and E N G M equal.

THEOREM.

Find the perimeters of the parallelograms L N M K and *m* N M *n*, and prove that they are unequal, and yet, that the parallelograms are equal in superficial area.

Let E A, which is a radius of, and common to, the circles X Y and Z, = 1. Then :

L K = N M, for they are diameters of the circles X Y and Z, of which E A is the radius, and common to both: and L K and N M are parallel to each other; therefore, L K and N M = 2 (E A) = 2. But, E A is common to the parallelograms E A N L and E A M K; therefore, L N and K M = 1; therefore, (L K + N M + L N + K M) = (2 + 2 + 1 + 1) = 6 = the perimeter of the parallelogram L N M K.

L N = K M = E A = 1, and the triangles N *m* L and M *n* K are similar right-angled triangles; but, it is neither axiomatic, nor self-evident, that the sides that contain the right angle in these triangles are in the ratio of 7 to 24: and yet, this can readily be demonstrated. By hypothesis, let $\frac{7}{25}$ (L N), or, $\frac{7}{25}$ (K M) $= \frac{7 \times 1}{25} = \cdot 28$ = L *m* and K *n*: and, $\frac{24}{25}$ (L N), or, $\frac{24}{25}$ (K M) $= \frac{24 \times 1}{25} = \cdot 96$ = *m* N and *n* M. Then: $(m\mathrm{N}^2 + \mathrm{L}m^2) = (n\mathrm{M}^2 + \mathrm{K}n^2)$; that is, $\cdot 28^2 + \cdot 96^2 = (\cdot 0784 + \cdot 9216) = 1 = \mathrm{LN}^2$ and KM^2; therefore, $\sqrt{1} = 1$ = L N and K M, which are equal to E A. Take another proof. With *n* as centre and *n* K as interval describe a circle. The circumference of this circle will cut the line *n* M at a point, say *x*. Join K*x*. (I have not made this addition, to avoid confusion in the diagram). Then: $(\mathrm{K}n^2 + nx^2) = \cdot 28^2 + \cdot 28^2 = (\cdot 0784 + \cdot 0784) = \cdot 1568 = \mathrm{K}x^2$: and, $(n\mathrm{M} - nx) = (\cdot 96 - \cdot 28) = \cdot 68 = x\mathrm{M}$; therefore, $\cdot 68^2 = \cdot 4624 = x\mathrm{M}^2$. But, K *n* M is a right-angled triangle, and K*x*M, a part of it, an oblique-angled triangle; and by Euclid: Prop. 12 : Book 2 : $\mathrm{K}x^2 + x\mathrm{M}^2 + 2(x\mathrm{M} \times xn,) = \mathrm{KM}^2$; that is, $\cdot 1568 + \cdot 4624 + 2(\cdot 68 \times \cdot 28) = \mathrm{KM}^2$: or, $(\cdot 1568 + \cdot 4624 + \cdot 3808) = 1 = \mathrm{KM}^2$; therefore, $\sqrt{1^2} = 1$ = K M. Now, *m n* = N M, and *m* N = *n* M, for they are opposite sides of the parallelogram *m* N M *n*, and N M is a diameter of the circle Z = 2, and *n* M = ·96; therefore, (N M + *m n* + *m* N + *n* M) = (2 + 2 + ·96 + ·96) = 5·92 = the perimeter of the parallelogram *m* N M *n*. Therefore, the perimeters of the parallelograms L N M K and *m* N M *n* are unequal. Q.E.D.

Again: M *n* E is a right-angled triangle, and M K E a part of it, is an oblique-angled triangle: and by Euclid: Prop. 12 : Book 2 : $\{\mathrm{MK}^2 + \mathrm{KE}^2 + 2(\mathrm{KE} \times \mathrm{K}n)\} = \mathrm{EM}^2$; that is, $\{1^2 + 1^2 + 2(1 \times \cdot 28)\}$ EM^2; or, $(1^2 + 1^2 + \cdot 56) = 2\cdot 56 = \mathrm{EM}^2$; therefore, $\sqrt{2\cdot 56} = 1\cdot 6$ = E M. But, E M is bisected by K A, for they are the diagonals of the parallelogram E A M K ; therefore, $\frac{E\,M}{2} = \frac{1{\cdot}6}{2} =$ $\cdot 8 = E\,a$; and $E\,a\,A$ is a right-angled triangle ; therefore $(E\,A^2 - E\,a^2) = 1^2 - \cdot 8^2) = (1 - \cdot 64) = \cdot 36 = a\,A^2$; therefore, $\sqrt{\cdot 36} = \cdot 6 =$ $a\,A$, and the sides that contain the right angle are in the ratio of 3 to 4. But, the similar and equal parallelograms E A M K and E A N L are each divided into four right-angled triangles, similar and equal to the triangle $E\,a\,A$, and it is self-evident that the parallelograms E A M K and E A N L together make up the parallelogram L N M K. Hence: $4\,(E\,a \times a\,A) = (N\,M \times M\,n)$, that is, $4\,(\cdot 8 \times \cdot 6) = (2 \times \cdot 96)$ or, $(4 \times \cdot 48) = (2 \times \cdot 96) = 1{\cdot}92$: and it follows, that the parallelograms L N M K and m N M n are equal in superficial area. Q.E.D.

THEOREM.

Prove that the perimeters of the parallelograms L N M K and K M $t\,p$ are equal, and yet, that they are not equal in superficial area.

Because Kp and Mt are equal to K L and M N, and K M common to the parallelograms K M N L and K M $t\,p$, by construction ; the perimeters of the parallelograms are equal. Now, I have proved that the area of the parallelogram K M N L $= 1{\cdot}92$: when $E\,A = 1$: but, $(K\,M \times M\,t) = (1 \times 2) = 2 =$ area of the parallelogram K M $t\,p$: and it follows, that the parallelograms K M N L and K M $p\,t$ are not equal in superficial area, although their perimeters are equal. Q.E.D.

These are some of the effects produced by "*difference of direction*," and "*quantity of turning*," in constructive geometry. In the search after π,—on what is called the exhaustive theory—by means of polygons, there is "*difference of direction*," and a "*quantity of turning*," involved at every doubling of the sides of the polygons : and we are beset with incommensurables at every step, whether we make an inscribed square, or an inscribed equilateral triangle, to a circle of radius 1, the "*initial position*," or starting point. Mathematicians not observing these facts, or, if observing them, failing to see the consequences, are led into the absurd fallacy, that they arrive at a close approximation to the true circumference of a circle of diameter unity, by their methods of computation.

Now, referring to the enclosed diagram *(Diagram XII.)*, the triangle E A D is a right-angled triangle, and the sides E A and A D, which contain the right angle, are in the ratio of 4 to 3, by construction. But, A *a* is a straight line, drawn from the right angle A perpendicular to the opposite side E D. And, it will be observed, that the line A *a* produced, that is, the line A K, passes through the point of intersection between the line E D and the circumference of the circle Y Z. Hence: the triangles E A *a* and A *a* D on each side of A *a*, are similar to the whole triangle E A D, and to each other. But, the triangles E A D, F K A, and E M G are similar right-angled triangles; and it follows, that if, in the triangle F K A, a straight line be drawn from the right angle K, perpendicular to the opposite side F A, the triangles on each side of this line will be similar to the whole triangle F K A, and to each other. And similarly with regard to the right-angled triangle E M G. But, it will be observed, that if straight lines be drawn from the angles K and M, in the triangles F K A and E M G, perpendicular to the opposite sides F A and E G, these lines will be parallel to A B, and perpendicular to A C, the generating lines of the diagram. Now, K F L and N G M are similar right-angled triangles, and similar to the right-angled triangle E A D; but, if straight lines be drawn from the right angles F and G, perpendicular to the opposite sides L K and N M, these lines will be neither perpendicular nor parallel to A B and A C, the generating lines of the diagram.

Now, it may be admitted, that the diagonal of a square is incommensurable with the sides; and from this *admitted* fact, Mathematicians argue that there is nothing irrational in supposing the circumference of a circle to be incommensurable with the diameter. How many times have such questions as the following been put to me? Is not the diagonal of a square incommensurable with its sides? Why, then, should it be thought irrational that the circumference of a circle is incommensurable with its diameter? The Mr. R—— referred to in the early part of my pamphlet "*Euclid at Fault*"—and he is a "*recognised Mathematician*"—in one of his communications said:—"*There is nothing irrational a priori in the admission that* π *may be indeterminate; the irrationality is in capriciously or arbitrarily fixing its nature before* WE FIND IT:" and in another communication he observed:—"*Unless first prin-*

ciples are well established—proved beyond question, if not axiomatic or self-evident—no discussion can be worth anything, or possess the least interest, to sincere and intelligent men. I am perfectly sure that Mr. Smith will admit, that if π cannot be shewn to be a determinate quantity, and shewn by a priori reasoning, that is, without reference to its arithmetical value—that process of reasoning by which he some time ago said he arrived at his conviction that $\pi = 3\frac{1}{8}$ cannot be valid. That π is determinate is, I say, a first principle in that process. Now, I again question the truth of that principle—or rather, proposition. As frequently I have said, it is not self-evident; and I know of no way in which it can be proved a priori. Surely it cannot be proved by practical geometry, or by calculations. It must be established by some kind of a priori and abstract reasoning: because it is brought in by Mr. Smith to find the arithmetical value of π. Now, I humbly submit that all Mr. Smith can say is away from the point, until he meet this claim I again make, namely, that he show how we must believe π to be a determinate quantity."* Mr. R——'s ideas are very prettily expressed: but, do they not amount to this, that the arithmetical value of π must be discovered, without reference to that "*indispensable instrument of science, Arithmetic,*" upon which all Mathematics are founded? It is essential that I should expose the fallacy and absurdity of such like reasoning, if reasoning it can be called.

Well, then, I have proved, with reference to the enclosed diagram, that when E A, the radius of the circles X Y and Z, = 1, the areas of the parallelograms L N M K and m N M n = 1·92: and, the area of the rectangular parallelogram K M $t p$ = 2: and, by analogy or proportion, 1·92 : 2 :: 3 : 3·125. But, I have also proved, that when E A = 1, the perimeters of the parallelograms K M N L and K M $t p$ = 6; and it is admitted by Mathematicians, that 6 (E A) = (6 × 1) = 6 = the perimeter of a regular inscribed hexagon to the circles X Y and Z: and, by analogy or proportion, 1·92 : 2 :: 6 : 6·25. But, 6 (radius × semi-radius) = area of a regular inscribed dodecagon to every circle;

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* I would ask the reader to mark the contrast between Mr. R——'s reasoning, and the assertion of the Rev. Professor Whitworth, in his original communication to Mr. James Smith.

therefore, $6\left(EA \times \frac{EA}{2}\right) = 6\,(1 \times \cdot 5) = 3$ = area of an inscribed regular dodecagon to the circles X Y and Z, when E A the radius of the circles = ·1 : and, by analogy or proportion, $6 : 6 \cdot 25 :: 3 : 3 \cdot 125$, and is similar to the analogy, $1 \cdot 92 : 2 :: 3 : 3 \cdot 125$. Now, if the THEORY that 8 circumferences of a circle = 25 diameters—which makes $\frac{25}{8} = 3 \cdot 125$ the arithmetical value of π—be a sound theory, it follows of necessity, that the area of the parallelogram L N M K or m N M n : the area of the parallelogram K M tp : : the area of a regular inscribed dodecagon to the circles X Y and Z : the area of the circles X Y and Z.

The diagonal of a square of which the sides = 1 is $\sqrt{2}$. This determinate arithmetical expression has always been a "*delusion and a snare*" to Mathematicians, and is continually being employed by them after a certain fashion, to prove—as they fancy—that the circumference and area of a circle are incommensurable with the diameter. A living authority on the "*Quadrature of the Circle*," (Mr. J. R. Young), observes:—"*The problem of* SQUARING THE CIRCLE, *as it is popularly called, has a twofold meaning—namely, the* GEOMETRICAL *quadrature, and the* NUMERICAL *quadrature.*" He proves, by a sound process of reasoning, that a square equivalent to a given circle exists, and then says:—"*It is plain, therefore, that there is nothing visionary or absurd in the search after this square, as if it were a thing that had no existence; although some very able Geometers have, strangely enough, condemned the enquiry on these grounds. The only sound reasons for abandoning the investigation are these two, namely—first, that the problem has been earnestly and laboriously attempted, by the profoundest Geometers, for thousands of years, and* THEY *have been obliged to abandon it in despair; and secondly, that the successful solution of it would be of no theoretical or practical value if furnished. As far as utility is concerned, the other form of the problem of the quadrature of the circle is by far the more important; that is, to discover the numerical measure of the surface of a circle from the measured length of its diameter being given. But, under this aspect of it, the accurate solution of the problem is really impracticable; it can be proved to be so. It is just as impracticable as it is to assign accurately the square root of* 2; *and, in fact, this square root* *does repeatedly enter into the approximative numerical process.*" Such are the publicly recorded opinions of Mr. J. Radford Young on the QUADRATURE OF THE CIRCLE.

Now, referring to the enclosed diagram *(Diagram XII.)*, let E A, the radius of, and common to, the circles X Y and Z $= \sqrt{2}$. Then: K M = E A, and K M is a side of, and common to, the parallelograms K M N L and K M *t p;* therefore, K M $= \sqrt{2}$: and since 6 (radius × semi-radius) = area of an inscribed regular dodecagon to every circle, it follows of necessity, that $6\left(KM \times \frac{KM}{2}\right) = 6\left(\sqrt{2} \times \frac{\sqrt{2}}{2}\right) = 6(\sqrt{2} \times \sqrt{\cdot 5}) = 6(\sqrt{2 \times \cdot 5}) = 6(\sqrt{1}) = 6 =$ area of an inscribed regular dodecagon to the circles X Y and Z. And, by analogy or proportion, 6 : 6·25 : : 3 : 3·125 ; and is similar to the analogy 3 : 3·125 : : 1·92 : 2 : and again demonstrates, that the area of the parallelogram L N M K or *m* N M *n*, is to the area of the rectangular parallelogram K M *t p*, as the area of a regular dodecagon to the area of its circumscribing circle.

Again: 2π (radius) = circumference in every circle; and, (circumference × semi-radius) = area in every circle. Now, $2\pi(EA) = 6{\cdot}25(\sqrt{2}) = \sqrt{6{\cdot}25^2 \times 2} = \sqrt{39{\cdot}0625 \times 2} = \sqrt{78{\cdot}125}$ = the circumference of the circles X Y and Z; and, $\frac{1}{2}(EA) = \frac{1}{2}(\sqrt{2}) = \sqrt{\cdot 5}$ = semi-radius of the circles; therefore, $(\sqrt{78{\cdot}125} \times \sqrt{\cdot 5}) = \sqrt{78{\cdot}125 \times \cdot 5}) = \sqrt{39{\cdot}0625} = 6{\cdot}25$ = area of the circles X Y and Z.

Again: the triangle E A D is a right-angled triangle, of which the sides that contain the right angle are in the ratio of 3 to 4, by construction. Now, when the side E A $= \sqrt{2}$, then, $\frac{3}{4}(EA) = \frac{3}{4}(\sqrt{2}) = \sqrt{\frac{3^2}{4^2} \times 2} = \sqrt{\frac{9}{16} \times 2} = \sqrt{\cdot 5625 \times 2} = \sqrt{1{\cdot}125}$ = the side A D, and, $\frac{5}{4}(EA) = \frac{5}{4}(\sqrt{2}) = \sqrt{\frac{5^2}{4^2} \times 2} = \sqrt{\frac{25}{16} \times 2} = \sqrt{1{\cdot}5625 \times 2} = \sqrt{3{\cdot}125}$ = the side E D; therefore, $(EA^2 + AD^2 + ED^2) = 3{\cdot}125\,(EA^2)$, that is, $(2 + 1{\cdot}125 + 3{\cdot}125) = (3{\cdot}125 \times 2) = 6{\cdot}25$: and this equation or identity = area of the circles X Y and Z, when the radius of the circle $= \sqrt{2}$.

You have told *me* that my reasoning is "*illogical and unsound;*" but I may now tell *you*, that the foregoing facts establish the truth of THEORY, that 8 circumferences = 25 diameters in every circle, making $\frac{25}{8} = 3{\cdot}125$, the true arithmetical value of the circumference of a circle of diameter unity; and $\frac{1}{3{\cdot}125}$ the true expression of the ratio between the diameter and circumference in every circle. To controvert this THEORY, you must demonstrate the following things to be untrue, and, by doing so, you will prove that I can set up no claim to be a "*reasoning geometrical investigator;*" and this you have brought yourself under an obligation to do, as "*a gentleman,*" a "*man of honour,*" a "*Christian minister,*" and a fair controversialist.

First: $(EA^2 + AD^2 + ED^2) = (PA^2 + AB^2)$, and this equation or identity is equal to the area of the square on P B, and it follows of necessity, that the square F B V T and the circles X Y and Z are exactly equal in superficial area.

Second: The sum of the squares of the three sides of the right-angled triangle K F L or N G M is equal to four times the area of the rectangle K M *t p*. It is self-evident that the triangles K F L and N G M are equal to half the rectangles F L A K and E N G M.

Third: The sum of the areas of the right-angled triangles K F L and N G M is equal to the sum of the areas of the parallelograms L N M K and *m* N M *n;* and it follows, that the rectangles F L A K and E N G M, and the parallelograms L N M K and *m* N M *n*, are equal in superficial area. But, the sum of the squares of three sides of the right-angled triangles K F L and N G M are equal to 3⅛ times the area of a square on the longer of the sides that contain the right angle, and = 8 when E A the radius of the circles X Y and Z = 1. But, $\pi\,(r^2)$ = area in every circle, and it follows, that $\sqrt{\frac{\text{area}}{\pi}}$ = radius in every circle. Now, $\sqrt{\frac{8}{3{\cdot}125}} = \sqrt{2{\cdot}56} = 1{\cdot}6$ = the longer of the sides that contain the right angle, in the triangles K F L and N G M; and, $\frac{3}{4}(1{\cdot}6) = \frac{3 \times 1{\cdot}6}{4} = 1{\cdot}2$ = the shorter of the sides that contain the right angle; and it follows, that $(1{\cdot}6 \times 1{\cdot}2) = 1{\cdot}92$ = area of the parallelograms F L A K and E N G M, and is equal to the area of the parallelograms L N M K and *m* N M *n*, when E A the radius of the circles X Y and Z = 1.

Fourth: The area of the irregular hexagon F L N G M K is equal to the sum of the areas of the parallelograms L N M K and *m* N M *n*.

Fifth: The triangles N *m* L, N *n* K, and P A R are similar right-angled triangles, and the longer of the sides that contain the right angle, is to the hypothenuse, as the perimeter of a regular hexagon to the circumference of its circumscribing circle.

Sixth: P R, the hypothenuse of the right-angled triangle, P A R, is exactly equal to the circumference of the circles X Y and Z: and D R, the base of the triangle, is exactly equal to the perimeter of a regular inscribed hexagon to the circles X Y and Z.

Now, when E A, the radius of the circles X Y and Z, = 1, A B, the base of the right-angled triangle P A B, $= \frac{EA}{4} = \frac{1}{4} = \cdot 25$: and, 7 (A B) $= (7 \times \cdot 25) = 1\cdot 75 =$ P A; therefore, $(PA^2 + AB^2) = (1\cdot 75^2 + \cdot 25^2)\ (3\cdot 0625 + \cdot 0625) = 3\cdot 125 = PB^2$; therefore, $\sqrt{3\cdot 125} = 1\cdot 767766 \ldots =$ P B. Will you dare to tell me that ·25 is not the Sine of the angle A P B when E A the radius of the circles X Y and Z = 1? Well, then, $\frac{AB}{PB} = \frac{\cdot 25}{1\cdot 767766} = \cdot 1414214$ is the trigonometrical Sine of the angle A P B: and, $\frac{PA}{PB} = \frac{1\cdot 75}{1\cdot 767766} = \cdot 9899500$ is the trigonometrical Cosine of the angle A P B. It is self-evident that we may make the *natural* or *geometrical* sine and cosine anything we please: and it will be a varying quantity according to the arithmetical value we may put upon E A, the radius of the circles X Y and Z. If we make E A = 4, then, P A = 7, and A B = 1, and in this case the *natural* or *geometrical* sine of the angle A P B = 1. If we compute the trigonometrical sine and cosine of the angle A P B by these values of P A and A B, we find a slight difference in their values, making the former ·1414213 and the latter ·9899494. But these differences cannot "*upset*" the fact, that the trigonometrical sine of the angle A P B $= \sqrt{\cdot 02}$, which may readily be demonstrated by extending the number of decimals. Will you dare to tell me that the trigonometrical sine of similar angles can be a varying quantity under any circumstances? Well, then, ·1414214 is the trigonometrical sine of the angle A P B as near as it can be ascertained to 7 places of decimals. The Logarithm corresponding to the natural number ·1414214 is 9·1505151, and this is the trigonometrical Log-sine of the angle A P B. The Logarithm corresponding to the natural number ·9899500 is 9·9956133, and this is the trigonometrical Log-cosine of the angle A P B.

Now, let P B the side subtending the right angle in the triangle P A B be any length, or, in other words, be represented by any finite arithmetical quantity, say 666; and be given to find the length of the sides P A and A B, and prove that they are in the ratio of 7 to 1.

Then:

As Sin. of angle A = Sin. of 90°	Log	10·0000000
: the given side P B = 666	Log.	2·8234742
: : Sin of angle A P B = Sin. of 8° 8′	Log.	9·1505151
		11·9739893
: the required side A B		10·0000000
= 666 × ·1414214 = 94·1866524	Log.	1·9739893

Again:

As Sin. of angle A = Sin. of 90°	Log.	10·0000000
: the given side P B = 666	Log.	2·8234742
: : Sin. of angle P B A = Sin. 81° 52′	Log.	9·9956133
		12·8190875
: the required side P A		10·0000000
= 666 × ·9899500 = 659·3067	Log.	2·8190875

Now, 7 (94·1866524) = 659·3065668, and is slightly less than the computed length of P A = 659·3067. These slight differences arise from the fact that P A B is an incommensurable right-angled triangle. Hence: The computed value of the length of A B is slightly less than its true value: and the computed value of the length of P A rather greater than its true value.

If we make the computations by Hutton's Tables.

Then:

As Sin. of angle A = Sin. 90°	Log.	10·0000000
: the given side P B = 666	Log.	2·8234742
: : Sin. of angle A P B = Sin. of 8° 8′	Log.	9·1506864
		11·9741606
: the required side A B		10·0000000
= 94·224 nearly	Log.	1·9741606

Again :

As Sin. of angle A = Sin. 90°............................	Log.	10·0000000
: the given side P B = 666..................................	Log.	2·8234742
: : Sin. of angle P B A = Sin. 81° 52′.....................	Log.	9·9956095
		12·8190837
: the required side P A		10·0000000
= 659·3 nearly...	Log.	2·8190837

Now, 7 (94·224) = 659·568. This would make the side P A less than 7 times the length of the side A B, which is absurd. But further : What are called the natural sine and natural cosine of the angle A P B are given by Hutton as ·1414772 and ·9899415. Now, 666 (·1414772) = 94·22382152: and 666 (·9899415) = 659·301039: and again, on this shewing, Hutton makes the side P A less than 7 times the length of the side A B, which is absurd. Will you dare to tell me, that what are called the natural sines and cosines in mathematical tables, are not treated, and reasoned upon, by Mathematicians, as the trigonometrical sines and cosines of angles ? With very few exceptions, the natural sines and cosines of angles, are very different from the trigonometrical sines and cosines.

The triangle P A R is a commensurable right-angled triangle, and the sides P A and A R which contain the right angle are in the ratio 7 to 24, by construction. The angle P R A is equal to twice the angle A P B in the triangle P A B. It can hardly be necessary for me to shew the author of "Choice and Chance," a "*recognised Mathematician*," how to convince himself of this fact.

Well, then, let P R the side subtending the right angle in the triangle P A R be 666 miles in length, and be given to find the lengths of the sides P A and A R which contain the right angle, and prove that they are in the ratio of 7 to 24.

Now, let E A, which is the radius of, and common to, the circles X Y and Z = the mystic number 4. Then : (E A + A D) = (4 + 3) = 7 = P A : (E A — A D) = (4 — 3) = 1 = A B : 24 (A B) = (24 × 1) = 24 = A R ; therefore, $(PA^2 + AR^2) = (7^2 + 24^2) = (49 + 576) = 625 = PR^2$; therefore, $\sqrt{625} = 25 = PR$, the side subtending the right angle, in the triangle P A R. Then : $\frac{PA}{PR} = \frac{7}{25} = \cdot 28$

is the trigonometrical sine of the angle P R A: and, $\frac{A\,R}{P\,R} = \frac{24}{25} =$ ·96 is the trigonometrical cosine of the angle P R A. The Logarithm corresponding to the natural number ·28 is 9·4471580: and the Logarithm corresponding to the natural number ·96 is 9·9822712: and these are the trigonometrical Log-sin. and Log-cosine of the angle P R A.

Then:

As Sin. of angle A = Sin. 90°..........................Log. 10·0000000
: the given side P R = 666 miles ....Log. 2·8234742
: : Sin. of angle P R A = Sin. 16° 16′....................Log. 9·4471580

12·2706322
: the required side P A .... 10·0000000

= 666 × ·28 = 186·48 miles..............................Log. 2·2706322

Again:

As Sin. of angle A = Sin. 90° ....Log. 10·0000000
: the given side P R = 666 miles ....Log. 2·8234742
: : Sin of angle R P A = Sin. 73° 44′ ....Log. 9·9822712

12·8057454
: the required side A R .... 10·0000000

= 666 × ·96 = 639·36 miles..............................Log. 2·8057454

Hence, by analogy or proportion:

P A : A R : : 7 : 24 ; that is, 186·48 : 632·36 : : 7 : 24.
P A : P R : : 7 : 25 ; that is, 186·48 : 666 : : 7 : 25.
And, A R : P R : : 24 : 25 ; that is, 639·36 : 666 : : 24 : 25.

And it follows of necessity, that $(\cdot 28^2 + \cdot 96^2) = (\cdot 0784 + \cdot 9215)$ $= 1 =$ unity; and meets the requirement of the trigonometrical axiom, Sin.^{2} + Cos.^{2} = unity, in every right-angled triangle: and, $(P\,A^2 + A\,R^2) = (186\cdot48^2 + 639\cdot36^2) = (34774\cdot7904 + 408781\cdot2096) =$ $443556 = P\,R^2$; therefore, $\sqrt{443556} = 666 = P\,R$.

If we make the computations by Hutton's Tables—

Then :

As Sin of angle A = Sin. 90°	Log.	10·0000000
: the given side P R = 666 miles	Log.	2·8234742
: : Sin. of angle P R A = Sin. 16° 16′	Log.	9·4473259
		12·2708001
: the required side P A		10·0000000
= 186·55 nearly	Log.	2·2708001

Again :

As Sin. of angle A = Sin. 90°	Log.	10·0000000
: the given side P R = 666 miles	Log.	2·8234742
: : Sin: of Angle R P A = Sin. 73° 44′	Log.	9·9825506
		12·8060248
: the required side A R		10·0000000
= 639·77 nearly	Log.	2·8060248

But, by analogy or proportion, 7 : 24 : : 186·55 : 639·46, &c. Thus, by the computations made from Hutton's Tables, the known and indisputable ratios of side to side, by construction, in the commensurable right-angled triangle P A R, are destroyed.

Well, then, the following is the irrefragable conclusion: The acute angle A P B in the right-angled triangle P A B, is an angle of 8° 8′. Hence: the angle P B A is an angle of 81° 52′: the angle P B R is an angle of 98° 8′: the angle A P R is an angle of 73° 44′: the angle P R A is an angle of 16° 16′: the angle P A R is a right angle = 90° : and, the sum of these angles is equal to 4 right angles. This is in perfect harmony with the THEORY that 8 circumferences = 25 diameters in every circle, making $\frac{25}{8}$ = 3·125, the true arithmetical value of the circumference of a circle of diameter unity, and $\frac{1}{3\cdot125}$ the true expression of the ratio between the diameter and circumference in every circle.

Now, Sir, unless you prove that my data is unsound, my arguments fallacious, and my reasoning illogical ; and so, vitiate my conclusions, and prove them absurd ; the inference will be, that I have "*brought down two birds at one shot*," a "*recognised Mathematician*," and a "*recognised*" computer of Mathematical Tables.

I may, or may not, have to trouble you with another communication; at any rate, it will not be for some time. My next literary and scientific labour, will take the form of a printed Letter to Professor Stokes, the *Mathematical* President elect of "*The British Association for the Advancement of Science.*"*

Faithfully yours,

THE REV. PROFESSOR WHITWORTH. JAMES SMITH.

Euclid's Elements of Plane Geometry, by W. D. Cooley, A.B., is a well known modern text-book for teaching the rudiments of Geometry, and has been highly commended by our leading Scientific Journals. In his Appendix to the fifth and sixth books, Mr. Cooley observes:—"*The same proposition (Prop.* 13*: Book* 6*) which enables us to find a mean proportional between two given lines, will also enable us to find a mean proportional between the first and second, and between the second and third; and thus to interpolate mean proportionals between the terms to any extent. But to find two mean proportionals, or A and B being given to find x and y, so that $A : x : : y : B$ is a problem beyond the reach of Plane Geometry.*"

With reference to this quotation, the Rev. Professor Whitworth makes some extraordinary statements in his Letter of Dec. 2, 1868. (*See page* 52). In a Letter not addressed to him at all, but of which I gave him a copy in my communication of the 30th Nov., I have referred to this quotation. *(See page* 47.*)* The Professor first says:—"*If your inverted commas mark a true quotation from Mr. Cooley's work, of course he must be quite wrong:*" and immediately adds:—"*The problem (as you propose it)*

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* The Reverend Professor Whitworth returned no more of my Letters, after that of the 18th January, 1868.

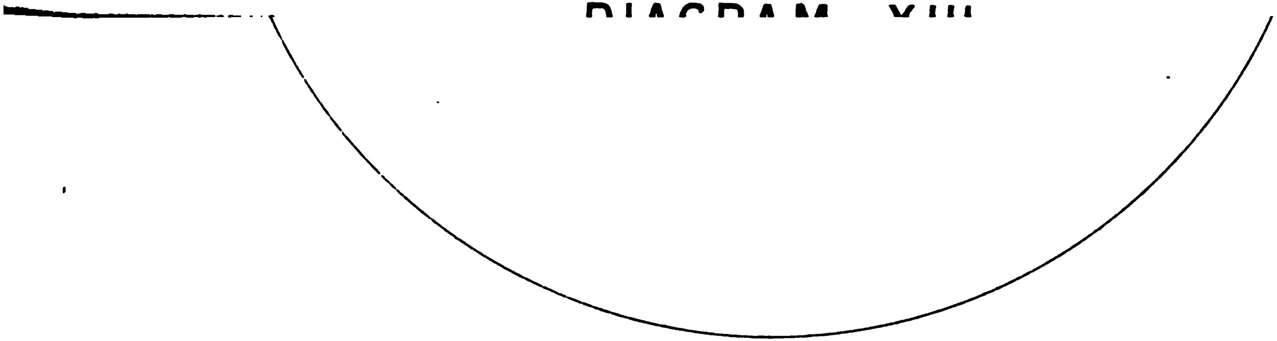
is not beyond the scope of pure Geometry, but is 'indeterminate;' that is, it admits of an infinite number of solutions." I did not quote from memory, as Professor Whitworth supposes, and there is no *lapsus* in my quotation, and it follows, that Professor Whitworth is absurdly inconsistent. I might refer to other absurdities in his communication of Dec. 2, 1868, but any reflective reader of that Letter will discover them without my assistance.

There are many ways of solving the problem, which Mr. Cooley asserts is beyond the reach of Plane Geometry, but my book is already big enough, and I shall content myself with directing your attention to one very remarkable solution.

CONSTRUCTION OF DIAGRAM XIII.

From a point A draw a straight line A C, and produce it to O, making C O equal to A C. With O as centre and O A as interval, describe the circle. Produce A O to meet and terminate in the circumference of the circle at the point B. From A B, which is obviously a diameter of the circle, cut off a part Ay, making Ay equal to $\frac{25}{32}$ (A B). From the point y draw a straight line perpendicular to A B to meet and terminate in the circumference of the circle at the point D, and join D A and D B, and thus construct the right-angled triangle A D B. From Ay cut off a part A E, making A E equal to $\frac{24}{25}$ (A y), and join D E; or, bisect O B at E, and join D E; and so construct the right-angled triangle D y E. From A O cut off a part A F, making A F equal to $\frac{4}{5}$ (A O), and from A F cut off a part A x, making A x equal to $\frac{4}{5}$ (A F). From the points O, F, and C, draw straight lines perpendicular to A B, to meet and terminate in the line D A, the hypothenuse of the right-triangle A D y, at the points

DIAGRAM XIII



G, H, and M. Produce O G to meet and terminate in the circumference of the circle at the point L. From L O cut off a part, N O, making N O equal to ¾ (O A), and join N A.

Let A denote the length of the line A C : x denote the length of the line A x : y denote the length of the line A y : and B denote the length of the line A B. Then: A : x :: y : B, and solves the problem, which, according to Mr. Cooley, "*is beyond the reach of Plane Geometry*;" but, which, according to the Reverend Professor Whitworth, "*admits of an infinite number of solutions.*" This learned Professor and "*recognised Mathematician*," is quite right; but, can you, Sir, tell me what he means by the "*indeterminate*" solution of a problem? Do you think he can give a *determinate* solution of Mr. Cooley's problem?

Let O denote the radius of the circle; and, by hypothesis, let A, which denotes the length of the line A C a semi-radius of the circle, by construction, = 4. Then: By computation, x = 5·12 : y = 12·5 : and B = 16: therefore, A : x : : y : B; that is, 4 : 5·12 : : 12·5 : 16. The product of the means is equal to the product of the extremes; that is, $(x \times y) = (\text{A} \times \text{B})$; or, $(5{\cdot}12 \times 12{\cdot}5) = (4 \times 16) = 64$; and it follows, that $\sqrt{x \times y} = \sqrt{\text{A} \times \text{B}}$, and that this equation or identity = O; that is to say, is equal to the radius of the circle: and we necessarily arrive at this result whatever arithmetical value we may put upon A. Hence: It is not *a problem beyond the reach of Plane Geometry, to find two mean proportionals, when A and B are given to find x and y, so that A : x : : y : B.*

In this example, A B is the diameter of a circle of which A C is the semi-radius. Hence: When A and B are given, making B equal to 4 times A, the solution of Mr. Cooley's problem is extremely simple.

By Euclid: Prop. 2: Book 2: the sum of the areas of the rectangles $AB \cdot AC$ and $AB \cdot CB = AB^2$; and is equal to the area of a circumscribing square to the circle. But, $(AB \times AC) = (OA \times OB)$, and this equation or identity, is equal to the area of a square on the radius of the circle; and it is axiomatic, if not self-evident, that by no other division of the line A B into two parts, A C and C B, but that which makes A C to C B in the ratio of 1 to 4, can we get this equation or identity.

By Euclid: Prop. 31: Book 3: A D B is a right-angled triangle; and, by computation, the length of the line $AE = 12$; the length of the line $Ey = {\cdot}5$; and the length of the line $yB = 3{\cdot}5$; when the radius of the circle $= 8$. By Euclid: Prop. 35: Book 3: and Prop. 8: Book 6: $(Ay \times yB) = Dy^2$; that is, $(12{\cdot}5 \times 3{\cdot}5) = 43{\cdot}75 = Dy^2$; therefore, $Dy = \sqrt{43{\cdot}75}$: and by analogy or proportion, $Ay : Dy :: Dy : yB$; that is, $12{\cdot}5 : \sqrt{43{\cdot}75} :: \sqrt{43{\cdot}75} : 3{\cdot}5$: or, $\sqrt{12{\cdot}5^2} : \sqrt{43{\cdot}75} :: \sqrt{43{\cdot}75} : \sqrt{3{\cdot}5^2}$; that is, $\sqrt{156{\cdot}25} : \sqrt{43{\cdot}75} :: \sqrt{43{\cdot}75} : \sqrt{12{\cdot}25}$. By Euclid: Prop. 8: Book 6: and Prop. 13: Book 6: DyA and DyB are similar right-angled triangles, and similar to the whole triangle A D B; therefore, by Euclid: Prop. 47: Book 1: $(Dy^2 + Ay^2) = (\sqrt{43{\cdot}75^2} + 12{\cdot}5^2) = (43{\cdot}75 + 156{\cdot}25) = 200 = AD^2$; therefore, $AD = \sqrt{200}$: $Dy^2 + yB^2 = (\sqrt{43{\cdot}75^2} + 3{\cdot}5^2) = (43{\cdot}75 + 12{\cdot}25) = 56 = DB^2$; therefore, $DB = \sqrt{56}$. But, $(AD^2 + DB^2) = (\sqrt{200^2} + \sqrt{56^2}) = \sqrt{(200 + 56)} = \sqrt{256} = 16 = AB$: and it follows of necessity, that $(AD^2$ —

Ay^2) = ($DB^2 - yB$), and this equation or identity = Dy^2: that is, (200 — 156·25) = (56 — 12·25) = 43·75; and again proves that $Dy = \sqrt{43·75}$. Hence: the area of a square on A D = 200.

DyA is a right-angled triangle, and D E A a part of it, is an oblique-angled triangle; and by Euclid: Prop. 12: Book 2: $\{DE^2 + EA^2 + 2(EA \times Ey)\} = AD^2$. Proof: DyE is a right-angled triangle, and the side $Dy = \sqrt{43·75}$, and the side Ey = ·5, when the diameter of the circle = 16, and makes the side E A, in the oblique-angled triangle DEA, = 12. Now, ($Dy^2 + Ey^2$) = ($\sqrt{43·75}^2$ + ·5^2) = (43·75 + ·25) = 44 = DE^2; and it follows, that $\{DE^2 + EA^2 + 2(EA \times Ey)\}$ = area of a square on A D; that is, {44 + 144 + 2 (12 + ·5)} = (44 + 144 + 12) = 200. Thus, "*on all shewings*," the area of a square on A D = 200.

Now, O A is the radius of the circle, by construction: and $\pi (r^2)$ = area in every circle; and I have demonstrated that the area of a square on A D = 200, when O A = 8. By hypothesis, let π = 3·141592. Then: $\pi (OA^2) = \pi (8^2)$ = (3·141592 × 64) = 201·061888 = area of the circle, on the hypothesis that π = 3·141592. But, $\frac{\text{area of circle}}{\frac{1}{4}\pi}$ = area of a circumscribing square to the circle, whatever be the value of π; therefore, $\frac{201·061888}{\frac{1}{4}\pi}$ $\frac{201·061888}{·785398}$ = 256, and is equal to the area of a circumscribing square to the circle. Will you, Sir, or will any other "*recognised Mathematician*," venture to tell me, that this proves the arithmetical value of π to be 3·141592? It proves that the area of a circumscribing square to a circle, is equal to 4 times the area of a square on the radius of the circle, and any other hypothetical value of π will produce this result; but as to the value of π, if it prove anything, it proves the absurdity of the orthodox notion, that π is an indeterminate arithmetical quantity. It certainly shivers to atoms, the absurd definition of the terms "*finite and determinate,*" given by that "*recognised Mathematician,*" the Reverend Professor Whitworth.

But further: $(OA \times AC) = 8 \times 4 = 32$: and, $OA \times AB = 8 \times 16 = 128$. Hence: the mean proportional between $(OA \times AC)$ and $(OA \times AB) = \sqrt{32 \times 128} = \sqrt{4096} = 64$ = area of a square on the radius of the circle: and $3\frac{1}{8}(OA^2) = 200$ = area of a square on AD, the hypothenuse of the right-angled triangle $Dy A$; and it follows, that the area of a square on AD = area of the circle.

Now, Sir, it is *conceivable,* you might *assert* that this is not a proof *per se* that the true arithmetical value of π is $3\frac{1}{8}$; and I shall now proceed to give you a proof, that "no *reasoning geometrical investigator*" would think of disputing; and which no Mathematician can controvert.

NOA is a right-angled triangle, and the sides NO and OA, which contain the right angle, are in the ratio of 3 to 4, by construction. Hence: When OA, the radius of the circle, = 8, then, $\frac{3}{4}(8) = \frac{3 \times 8}{4} = 6$ = the side NO; and $\frac{5}{4}(8) = \frac{5 \times 8}{4} = 10$ = the side NA; therefore, $(NO^2 + OA^2 + NA^2) = 3\frac{1}{8}(OA^2)$; that is, $(6^2 + 8^2 + 10^2) = 3\frac{1}{8}(8^2)$; or, $(36 + 64 + 100) = (3{\cdot}125 \times 64)$; and this equation or identity = 200. But, I have demonstrated that the area of a square on AD, the hypothenuse of the right-angled triangle $Dy A$, = 200, when $OA = 8$; and it follows of necessity, that the sum of the squares of the three sides of the triangle N O A, is equal to the area of a square on A D, whatever arithmetical value we may put upon the radius of the circle. Now, $\frac{NO^2 + OA^2 + NA^2}{\frac{1}{4}(3\frac{1}{8})} = \frac{200}{\cdot 78125} = 256 =$ area of a circumscribing square to the circle; and fixes $\frac{25}{8} = 3\cdot 125$ as the true arithmetical value of π, and establishes the truth of the THEORY, that 8 circumferences = 25 diameters in every circle. Q. E. D.

If Professor de Morgan attempt to controvert this conclusion, where will he be? To quote his own words, I may answer:—"*Overhead, in a cloud, on one stick laid across two others, under a nimbus of* 3·14159265 . . . *diameters to the circumference.*" But, he certainly will not be in "*π glory.*" "*Oh for a drawing of this scene.*"*

Now, although 1 and $\sqrt{1}$ are definite expressions, and *abstractly* of equal arithmetical value, their mathematical functions are very different indeed. For example: $\frac{3}{5}(1) = \cdot 6$, and $\frac{4}{5}(1) = \cdot 8$; therefore, $(\cdot 6^2 + \cdot 8^2) = (\cdot 36 + \cdot 64) =$ unity. Hence: ·6 to ·8 : ·6 to 1 : and ·8 to 1, express the ratios of side to side in the right-angled triangle N O A, whatever arithmetical value we may put upon the line D A, the radius of the circle. But, $\frac{3}{5}(\sqrt{1}) = \sqrt{\frac{3 \times 1}{5}} = \sqrt{\cdot 36} = \cdot 6$; and $\frac{4}{5}(\sqrt{1}) = \sqrt{\frac{4 \times 1}{5}} = \sqrt{\cdot 64} = \cdot 8$. Now, $(\sqrt{\cdot 6} + \sqrt{\cdot 8}) = (\sqrt{\cdot 6 + \cdot 8}) = \sqrt{1\cdot 4}$, and is greater than unity. Or, $(\sqrt{\cdot 6^2} + \sqrt{\cdot 8^2}) = (\cdot 6 + \cdot 8) = 1\cdot 4$, and is greater than unity. But, $(\sqrt{\cdot 6} \times \sqrt{\cdot 8}) = \sqrt{\cdot 6 \times \cdot 8} = \sqrt{\cdot 48}$, and is less than unity. It appears to me that Mathematicians entirely overlook these plain and simple truths, and utterly ignore them, in their application of Mathematics to Geometry.

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* See *Athenæum* : July 29 : 1865 : Article : A Budget of Paradoxes.

Let A denote the length of the line A C: x denote the length of the line A x: O denote the length of the line O A; that is, the length of the radius of the circle: let y denote the length of the line A y: and B denote the length of the line A B, that is, the length of the diameter of the circle.

I have given you the computed values of x and y, when A C = 4, and demonstrated that A : x :: y : B, and on any other value of A C we may make the computations, and prove that the solution of Mr. Cooley's problem is not beyond the reach of Plane Geometry. It is self-evident, that O = 8, when A C = 4, and since C O = A C, by construction; it follows, that $\sqrt{x \times y}$, $= \sqrt{A \times B}$, and both = O.

Now, Sir, suppose me to say, that I can put another value upon A, that is, upon the length of the line A C, compute the values of x, O, y and B, prove that A : x :: y : B : and yet, shew that the mean proportionals $\sqrt{x \times y}$ and $\sqrt{A \times B} =$ O, that is, are equal to the radius of the circle. You might say, and *apparently* with every shew of reason, that I am mad: and yet, "*tis not more strange than true*," that this is not an impossibility. If I am mad, I am not so far gone, that there is not a method in my madness. But, I maintain that I am neither chargeable with madness, nor with "*ignorance and folly*," "*but speak the words of truth and soberness.*"

Now for my proofs :—

Let $A = \sqrt{1}$. Then: $2(A) = 2(\sqrt{2}) = \sqrt{2^2 \times 2} = \sqrt{4 \times 2}$

$= \sqrt{8} = O$. $2(O) = 2(\sqrt{8}) = \sqrt{2^2 \times 8} = \sqrt{4 \times 8} = \sqrt{32} = B$. $\frac{25}{32}(B) = \frac{25}{32}(\sqrt{32}) = \sqrt{\left(\frac{25^2}{32^2} \times 32\right)} = \sqrt{\left(\frac{625}{1024} \times 32\right)} = \sqrt{({\cdot}6103515625 \times 32)} = \sqrt{19{\cdot}53125} = y$. *(At this point, I may direct your attention to the fact, that* $\cdot6103515625 = \left(\frac{\pi}{4}\right)^2 = \left(\frac{3{\cdot}125}{4}\right)^2 = {\cdot}78125^2$). $\frac{4}{5}(O) = \frac{4}{5}(\sqrt{8}) = \sqrt{\left(\frac{4^2}{5^2} \times 8\right)} = \sqrt{\left(\frac{16}{25} \times 8\right)} = \sqrt{{\cdot}64 \times 8} = \sqrt{5{\cdot}12}$ = the length of the line A F. $\frac{4}{5}$ (A F) $= \frac{4}{5}(\sqrt{5{\cdot}12}) = \sqrt{{\cdot}64 \times 5{\cdot}12} = \sqrt{3{\cdot}2768} = x$. Therefore, $A : x : : y : B$: that is, $\sqrt{2} : \sqrt{3{\cdot}2768} : : \sqrt{19{\cdot}53125} : \sqrt{32}$. The product of the extremes, is equal to the product of the means; that is to say, $(\sqrt{2} \times \sqrt{32}) = (\sqrt{3{\cdot}2768} \times \sqrt{19{\cdot}53125})$: or, $\sqrt{(2 \times 32)} = \sqrt{(3{\cdot}2768 \times 19{\cdot}53125)}$. But, this equation or identity $= \sqrt{64} = 8$, and is equal to the radius of the circle, when A C the semi-radius $= 4$. But further: $(A \times O) = (\sqrt{2} \times \sqrt{8}) = \sqrt{2 \times 8} = \sqrt{16}$: and, $(B \times O) = (\sqrt{32} \times \sqrt{8}) = \sqrt{32 \times 8} = \sqrt{256}$: and the mean proportional between $\sqrt{16}$ and $\sqrt{256} = \sqrt{16 \times 256} = \sqrt{4096} = 64$; and is equal to the area of a square on the radius of the circle, when A C the semi-radius $= 4$. There can be no effect without a cause; and, any honest Mathematician, if he be a "*reasoning geometrical investigator,*" will readly trace this effect to its true cause. It arises from the simple fact, that the double of the square root of any finite quantity, is not the square root of twice that quantity, but the square root of four times that quantity; and the result I have demonstrated follows of necessity.

Let that unscrupulous critic, Professor de Morgan, try his hand once more, (he has told the scientific world, through the columns of the *Athenæum*, that he hopes to have "*many a bit of sport*" with me in the future, as he has "*had in the past*,") and prove that James Smith, that "*most amusing of all blunderers*," and, *most good humoured of all thunderers*,"[*] has "*convicted himself of ignorance and folly, with an honesty and candour, worthy of a better value of* π."[†]

The diagram XIII. is a perfect study for Geometers. To exhaust the properties of this remarkable geometrical figure would require something more than a pamphlet; and I shall leave it to other Geometers to pursue the enquiry. But, I will direct your attention to one or two facts with reference to it, which are deserving of the attention of Section A of the "*British Association.*"

Produce the line G O to meet and terminate in the circumference of the circle at a point, say N. It is self-evident, that G N would be a longer line than G A. Now, conceive the line G N to revolve in the direction of A, and be arrested in the course of its revolution at the point F. It will then have passed along the line O A equal to one–fifth part of its length, and O A is the radius of the circle. Conceive the line G N to revolve again in the same direction, and be arrested at the point x. It will then have passed along the line O A a further distance, equal to one-fifth part of the length of the line A F. It is obvious that we might pursue this operation *mathematically, ad infinitum:* but we could never get to *zero;* that is to say, we could never get to the point A, or make G N coincide *geometrically* with G A.

Again: From the point A draw a straight line A K parallel to O L and at right angles to A B, and therefore tangental to the circle. Conceive the line A K to revolve continuously in the direction of D until it arrive at the point B. It is self-evident, that if arrested at this point, the line A K will coincide with the line A B. Now, we may conceive the line A K to be arrested in the course of its revolution, at a thousand points in the semi-circle A L D B, and straight lines drawn from these points to the point B: but, there is no other point but D in the semi-circle, from which we can let fall the perpendicular Dy, so that A : x :: y : B.

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[*] See *Athenæum*, Nov. 25, 1865.

[†] See *Athenæum*, June 24, 1865.

Now, there must be a ratio between the sides that contain the right angle in the triangles G O F, H Fx and M C A. Would it not be a rational and sensible enquiry for the "*guiding stars*" of Section A of the *British Association* to find these ratios? If they can find these ratios, it follows, that they may find the values of the angles. Other matters worth enquiring into with reference to this geometrical figure, cannot fail to suggest themselves to you.

This brings me to the part that *you*, Sir, have played in the "*upward path*" of my literary and scientific career. You have certainly not *beset* that path, after the fashion of a Whitworth "*dragon.*"

The British Association held its thirty-second meeting at Cambridge, in October 1862, and I again freely distributed a pamphlet amongst the assembled Members. I had made up my mind to read a Paper, if possible; and I thought that at this *hot-bed* of Mathematics, there would surely be found somebody ready and willing to have a "*passage at arms*" with me by correspondence, even if I failed to get a hearing in the Physical Section; but I calculated without my host, and the seed sown on that occasion produced no fruit. I sent a copy of the pamphlet to the President elect some time before the meeting (and a copy to you at the same time), and wrote him a Letter, enclosing the Paper I proposed to read. In my Letter I referred to the way I had been insulted at the Aberdeen meeting of the Association. The following was his reply:—

CAMBRIDGE, *Sept.* 26, 1862.

SIR,

I have the honour to acknowledge the receipt of your Letter, and printed Letter concerning the quadrature of the circle.

You are of course aware that all papers or communications, or at least the titles of them, must be submitted to the Secretary of the Section in which they are proposed to be read. You are also aware that a general rule excludes the subject of your paper, and I am sorry to say that there is no probability of this rule being waved upon the present occasion.

If I might venture to offer advice, it would be not to attempt to obtain a hearing. In any case, I am sure your request would not be received with insult; but I am sure it would be politely declined. The matter, however, properly belongs to the President of the Section in question and to its Secretary, to whom I beg to refer you, and am,

Sir,
Your obedient Servant,
R. WILLIS.

J. SMITH, Esq.

On receipt of this communication I immediately addressed a Letter to you, as the President elect of the Physical Section, enclosing copy of the Paper I proposed to read. I again wrote you on my arrival at Cambridge to attend the meeting, but to neither of my Letters did I ever receive any reply. Such a course of procedure, if not insulting, was at least discourteous, and little in harmony with the polite epistle of Professor Willis.

On the 31st March last, I addressed the following Letter to you :—

SIR,

In the enclosed Diagram (see Diagram III.) the triangles O B C, O B P, O C F, O P F, O F R, O F y, O y V, and O R V, are similar right-angled triangles, and have the sides that contain the right angle in the ratio of 4 to 3, by construction.

Hence :—

When O K the radius of the circle P = 2,

Then :

O B the radius of the circle X = 4.
O C the radius of the circle Y = 5.
O F the radius of the circle M = 6·25.
O R the radius of the circle X Z = 7·8125.

And the line O V, which is the hypothenuse of, and common to, the triangles O R V and O y V = $3{\cdot}125^2$ = 9·765625.

THEOREM.

Prove that the area of the circle X Z, is exactly equal to the sum of the squares of the four sides of the quadrilateral O y V R.

This Theorem can readily be solved; and as an old Life Member of the British Association for the Advancement of Science, I venture to put the following question to you as the President elect, and a "*recognised Mathematician.*" Is the solution of this Theorem a fit and proper subject for a Paper in Section A of the Association ? An answer will oblige.

I am, Sir,
Yours very respectfully,
JAMES SMITH.

G. G. STOKES, Esq., F.R.S., D.C.L., &c.,
Cambridge.

Now, Sir, you have chosen to treat that Letter with silent contempt. With the tone of it I hardly think you can find fault. Well, then, it appears to me, that whatever nearly seven years may have done in the way of improving your *scientific knowledge*, it has done nothing towards improving your *knowledge* of "*the usual courtesies of good society.*"

In bringing two such important matters under your notice, as "*Euclid inconsistent with himself*," and "*Mathematics, as applied to Geometry by Mathematicians, 'a mockery, delusion, and a snare,'*" I, as a Life Member of the British Association, have done my duty; and if you, Sir, as its President, do yours, I may hear you announce in your Presidential address, at the forthcoming meeting of that body, something equivalent to declaring, that J. M. Wilson, Esq., the Reverend Professor Whitworth, and one—the learned Professor de Morgan—who thinks himself a greater Mathematician than either of them, are all "*nailed by themselves to the barn-door, as the delegates of miscalculated and disorganised failure.*"*

Again: Sir, with reference to the diagram enclosed in my Letter of the 31st March last (see Diagram III.), I beg to direct your attention to the following facts:—

It is self-evident, that O P F C and O y V R are similar quadrilaterals, and it follows of necessity, that if the sum of the squares of the four sides of the quadrilateral O P F C is equal to the area of the circle Y, the sum of the squares of the four sides of the quadrilateral OyV R is equal to the area of the circle X Z. Now, when O K the radius of the circle P $= 2$, then, O C the radius of the circle Y $= 5 : \frac{3}{4}$ (O C) $= 3·75 =$ C F; and since O F is the hypothenuse of, and common to, the right-angled triangles O C F and O P F, it follows, that O P = O C, and P F = C F. Hence: $(OC^2 + CF^2 + OP^2 + PF^2) = 3\frac{1}{8}(OC^2)$, or, $3\frac{1}{8}(OP^2)$; that is, $(5^2 + 3{\cdot}75^2 + 5^2 + 3{\cdot}75^2) = 3\frac{1}{8}(5^2)$, or, $(25 + 14{\cdot}0625 + 25 + 14{\cdot}0625) = (3.125 \times 25)$, and this equation or identity $= 78{\cdot}125$, and is equal to the area of the circle Y. Any "*reasoning geometrical investigator*" may readily convince himself, that the circle Y and the square *m n o p* standing upon it, are exactly equal in superficial area.

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* See *Athenæum*: July 25, 1868: Article; *Our Library Table.*

THEOREM.

Let the area of the circle P be represented by any finite arithmetical quantity, say 60. Prove that the sum of the squares of the diagonals in the quadrilateral O *y* V R, together with four times the square of the line F *x* which joins the middle points of the diagonals, is equal to the sum of the squares of the four sides of the quadrilateral O *y* V R.

With *you*, Sir, it is quite unnecessary for me to work out this *theorem* in detail, and I shall content myself with giving you the values of the lines. The diagonal OV $= \sqrt{457{\cdot}763671875}$. The diagonal $yR = \sqrt{421{\cdot}875}$ · O F $= \sqrt{187{\cdot}5}$. O F is to F *x* in the ratio of 4 to ·875, by construction: therefore, $16 : {\cdot}765625 :: 187{\cdot}5 : 8{\cdot}972167968\bar{7}5$; and it follows, that the area of a square on the line F *x* = 8·97216796875. But, O R and $Oy = \sqrt{222{\cdot}96875}$, and yV and $RV = \sqrt{164{\cdot}794921875}$; therefore, $(OR^2 + RV^2 + Oy^2 + yV^2) = \{OV^2 + yR^2 + 4\,(Fx^2)\}$; that is, $\{292{\cdot}96875 + 164{\cdot}794921875 + 292{\cdot}96875 + 164{\cdot}794921875\} = \{457{\cdot}763671875 + 421{\cdot}875 + 35{\cdot}888171875\}$; and this equation or identity $= 915{\cdot}52734375$ = area of the circle X Z. Proof: O R the radius of the circle X Z $= \sqrt{292{\cdot}96875}$, and $\pi\ (r^2)$ = area in every circle; therefore, $\pi\ \sqrt{(292{\cdot}96875^2)} = (3{\cdot}125 \times 292{\cdot}96875) = 915{\cdot}52734375$ = area of the circle X Z. If I am right, in my demonstration of this *Theorem*, π can be nothing else but $\frac{25}{8} = 3{\cdot}125$; and if I am wrong, *you*, Sir, as a "*recognised Mathematician*," can surely prove it.

I shall now repeat the THEOREM given in my Letter of the 31st March last, in a somewhat different form; and then give you the solution of it, and connect it with the foregoing *Theorem* and its solution.

THEOREM.

Let the area of the circle P be represented by any finite quantity, say 60. Find the values of the sides of the quadrilateral O y V R, and prove that the sum of the squares of the four sides, is equal to the area of the circle X Z.

I must premise, that if *you*, Sir, were to *assert* that the solution of this *Theorem* is impossible, you would simply make an assertion that is untrue; and if you were to admit that you cannot solve this *Theorem*, you would forfeit all claim to the title of "*recognised Mathematician.*" But, I have no hesitation in saying, that you not only *know* that the solution of the *Theorem* is not impossible, but that you also *know* how to solve it. If I am wrong in this statement, you can prove it. Well, then, if you *know* how, and yet decline to solve the *Theorem* (you will have the opportunity of doing so at Exeter), can you be an *honest* advocate of scientific truth? Would you not stamp yourself as one of the De Morgan, Whitworth, and Wilson school of Scientific morality? Or, if you examine my solutions of the *Theorems* I have given you, and find them incontrovertible, how can you, *if an honest man*, hesitate to admit, that they demonstrate the truth of the THEORY that 8 circumferences = 25 diameters in every circle, and make $\frac{25}{8} = 3{\cdot}125$, the true arithmetical value of π?

Again: Referring to diagram III., I may observe, it is self-evident that with O as centre and O V as interval, we might describe another circle. Conceive this circle to be added to the diagram, and denoted by X Y Z.

Now, when O K = 2, then, $OV = (3\frac{1}{8})^2 = 9{\cdot}765625$, by construction: and $\pi\,(OV^2) = 3\frac{1}{8}\,(9{\cdot}765625^2) = (3{\cdot}125)^5$ $= 298{\cdot}023223876953125$. But, $3\frac{1}{8}\,(OK^2) = (3{\cdot}125 \times 4)$ $= 12{\cdot}5$: and, $\frac{298.023223876953125}{12{\cdot}5} = 23{\cdot}841857910015625$, and this is the number of times the area of the circle P is contained in the area of the circle X Y Z, when the area of the circle P = 60: therefore, $60\,(23{\cdot}841857910015625)$ $= 1430{\cdot}511474609375$ = area of the circle X Y Z. Now, since $\pi\,(r^2)$ = area in every circle, it follows of necessity, that $\sqrt{\frac{\text{area}}{\pi}}$ = radius in every circle: therefore, $\sqrt{\left(\frac{1430{\cdot}511474609375}{3{\cdot}125}\right)} = \sqrt{(457{\cdot}763671875)}$ = O V the radius of the circle X Y Z. But, the sides O R and R V which contain the right angle in the triangle O R V, are in the ratio of 3 to 4, and the sides O R and O V in the ratio of 4 to 5, by construction; therefore, $\frac{4}{5}\,(OV)$ $= \frac{4}{5}\,(\sqrt{457{\cdot}763671875}) = \sqrt{292{\cdot}96875}$ = O R: and, $\frac{3}{5}\,\sqrt{457{\cdot}763671875}) = \sqrt{164{\cdot}794921875}$. But, O y = O R, and y V = R V, in the quadrilateral O y V R; therefore, $(Oy^2 + yV^2 + OR^2 + RV) = 3\frac{1}{8}\,(Oy)$ or, $3\frac{1}{8}\,(OR^2)$; that is, $(292{\cdot}96875 + 164{\cdot}794921875 + 292{\cdot}96875 +$

$164·794921875) = (3·125 \times 292·96875)$, and this equation or identity = area of the circle X Z. Q. E. D.

In conclusion. This either *is*, or *is not*, a demonstration of the *Theorem*. If it *is*, *you*, Sir, as an honest man, ought to admit it. If it *is not*, surely as a "*recognised Mathematician*" you ought to prove it. If you decline to adopt one or other of these alternatives, and resolve to treat both with silent contempt, you will simply prove that, rather than admit the truth of my solution of this *Theorem*, you elect to take your place with De Morgan, Whitworth, and Wilson, in the ranks of that numerous class, "*who despise wisdom and instruction.*"

I am, Sir,

Yours respectfully,

JAMES SMITH.

APPENDIX A.

EUCLID AT FAULT.

To JOSEPH DALTON HOOKER, ESQ., F.R.S., D.C.L., *President Elect of the British Association for the Advancement of Science,* 1868—1869.

SIR,

I am a very old Life Member of "*The British Association for the Advancement of Science,*" and have reason to believe that I am better known, than respected, by the leading Members of the Mathematical and Physical Section. The Astronomer Royal, in his opening address to that Section, at the thirty-first Meeting of the Association, held in Manchester, in 1861, observed :—"*It was known to those present that great ingenuity had been employed upon certain abstract propositions of Mathematics which had been rejected by the learned in all ages, such as finding the length of the circle, and perpetual motion. In the best academies of Europe, it was estabilshed as a rule that subjects of that kind should not be admitted, and it was desirable that such communications should not be made to that Section, as they were a mere loss of time.*"* These remarks arose out of a small pamphlet I distributed among the mathematical Members at that Meeting, a copy of which I had taken care to put the Astronomer Royal in possession of, previous to giving his opening address.

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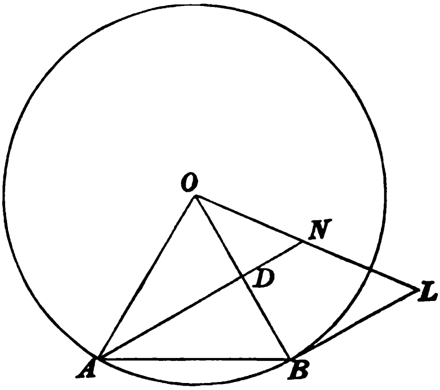
* Transactions of the British Association for 1861. Notices and Abstracts of Miscellaneous Communications to the Sections. Page 2.

Notwithstanding the rules adopted "*in the best academies of Europe,*" men—call them learned or call them unlearned—have not been prevented from *spending* their time—*wasting* it the Astronomer Royal would say—on such subjects as "*The Quadrature and Rectification of the Circle;*" and I am not ashamed to confess that I am among the number. My labours have led to the discovery of the remarkable fact, that *Euclid is at fault* in one of his most important Theorems; that is to say, that the eighth proposition of the sixth book of Euclid is not of general and universal application, and is therefore *not true* under all circumstances; and consequently, is inconsistent with the forty-seventh proposition of the first book. The proof of this fact is so plain and simple, as to be within the capacity of any man possessed of the most moderate geometrical and mathematical attainments; nay, I might say, within the capacity of any first class school-boy; and to *you*, Sir, my demonstrations will be as palpable, as that the square of 3 is 9.

I must now tell you how I was led to make this important discovery. I have for years attended regularly the Meetings of the British Association, but, not being allowed—by the rules of that body—to read a Paper in the Mathematical and Physical Section, on "*The Quadrature of the Circle,*" I have, from time to time, brought out pamphlets on the subject, and these I have freely distributed among the mathematical Members of the Association. At the last Meeting, in Dundee, I distributed one. At the time I was writing that Pamphlet, a Mr. and Mrs. S———, from Dumfriesshire, were on a visit to their son, resident in Liverpool; and being old friends of my wife's family, came out to Seaforth and made a call upon us. I happened to mention the fact to Mr. S———, that I was engaged in writing a Letter to His Grace the Duke of Buccleuch, the President elect of the British Association, and told him the subject of it; when Mr. S——— informed me that in early life, he had himself been a good Mathematician, and still took a deep interest in Mathematics. This led us into a conversation on the subject of *Squaring the Circle*, which resulted in my presenting him with several of my pamphlets. He then told me that his brother was an excellent Mathematician and a man of leisure; and, that he had a relative residing in the immediate neighbourhood of his brother, who was a first-class Mathematician, and that he should send the pamphlets to them, and induce those gentlemen to give them their careful attention, which his own health and business engagements would not admit of *his* doing. As soon as my Letter to the Duke of Buccleuch was published, I sent Mr. S—— copies, and in December last I received a communication from him, which led to a correspondence that would make a large volume; in which his relative, whom I may call Mr. R———, played the part of my chief opponent. Only some two or three communications passed between me and my friend's brother. Mr. S—— played the part, as it were, of a medium, or I might say referee; that is to say, my Letters were addressed to him, and after perusal, forwarded to his relative; and Mr. R———'s communications came to me through my friend, who really acted as referee, inasmuch as he kept Mr. R—— and me within the legitimate bounds of controversy.

In the course of the correspondence, I think I extracted from Mr. R——— every conceivable objection he could advance against the truth of the *Theory*, that 8 circumferences = 25 diameters in every circle; which makes $\frac{25}{8} = 3{\cdot}125$ the true arithmetical value of π; and, $\frac{1}{3{\cdot}125}$ the true expression of the ratio between the diameter and circumference, in every circle. I pointed out to Mr. R——— that in attempting to find the value of π, by multilateral-sided inscribed polygons to a circle, whether we make an inscribed equilateral triangle to the circle, or an inscribed square to the circle, our starting point, the ratio of chord to arc in every successive polygon, is a varying ratio; and I shewed him that the reason is plain enough. It follows from the fact, that the sides of every successive polygon are convergent and divergent lines from the sides of those that precede them; and consequently, that we can never, by these processes arrive at the value of π, or the true ratio of diameter to circumference in a circle: nor, can we arrive at them, by any other process, in which we attempt (directly) to measure a curved line by means of straight lines. Hence the inapplicability of the forty-seventh proposition of the first book of Euclid, to measure directly the circumference of a circle. I answered every objection started by Mr. R———, still he was not convinced; and it then occurred to me, that nothing short of proving the forty-seventh proposition of the first book, inconsistent with some other Theorem of Euclid, would ever convince a recognised Mathematician, that the arithmetical value of π is a finite and determinite quantity. But, whoever thought of questioning Euclid? Professor de Morgan never went further than attempt to prove Euclid illogical.[*] But, it never entered into ***his mathematical philosophy***, to dream of proving Euclid ***positively*** at fault. How then was it likely that I should ever think of doing so.

Towards the end of April I was called away to Scotland, and on my return home spent a few days at Windermere, and it was during my stay there that I made the important discovery, that *Euclid is at fault.* The morning of the 2nd May was very wet at Windermere, and it occurred to me—as I could not leave the Hotel—that I could not better pass the time than by writing a Letter to Mr. S——, enclosing a diagram, represented by the geometrical figure in the margin, in which the angle A and the sides O B and O L, in the triangles O A B and O B L, are bisected by the line A N. This I intended as introductory to a succession of diagrams, explanatory and demonstrative of the important discovery, that, ***the eighth proposition of the sixth book of Euclid is inconsistent with the forty-seventh proposition of the first book; and that it is the former, not the latter, that is at fault.***



It subsequently occurred to me, that if Euclid could be ***at fault*** in one Theorem, he might be ***at fault*** in others, and upon further

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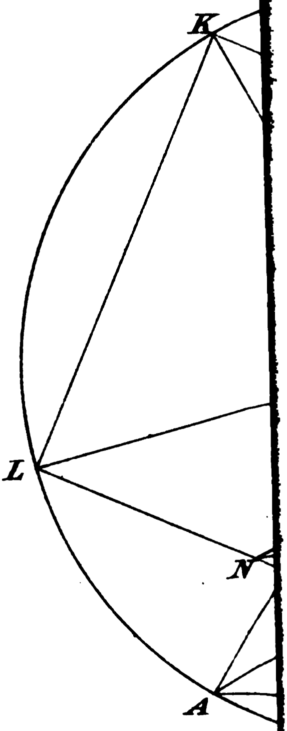
[*] Notes and Queries, 3rd S. VI. August 27, 1864. P. 161. Had the learned Professor asserted that the 18th and 19th Propositions of Euclid's Third Book are superfluous—what is proved by these propositions being established by the 16th Proposition—I should have agreed with him.


K
L
N
A

examination I discovered, that *the twelfth and thirteenth propositions of the second book of Euclid, are also inconsistent with the forty-seventh proposition of the first book, and again, that it is the former, not the latter, that is at fault.*

In the correspondence with my friend, I directed his attention to a succession of geometrical figures, derived from the foregoing very simple diagram, but for my present purpose it is quite sufficient to take one of them.

In the annexed diagram, (*Diagram XIV.*) let A and B be two points dotted at random. Join A B. On A B describe the equilateral triangle O A B, and from the angles of the triangle draw straight lines, bisecting the angles and their subtending sides at the points C, D, and E. With O as centre, and O A or O B as interval, describe the circle. Bisect E O at F, and E B at G. Join D G, and from the point F draw a straight line parallel to A E and D G, to meet and terminate in the circumference of the circle at the point H, and join F C. Produce B O to meet and terminate in the circumference of the circle at the point K, and join K H and H B, producing the right-angled triangle K H B. (*Euclid: Prop.* 31, *Book* 3.) From the angle B draw a straight line at right-angles to K B, and therefore tangental to the circle, to meet K H produced at M, constructing the right-angled triangle K B M. From the point H let fall the perpendicular H P. From the angle H draw a straight line through the point O, the centre of the circle, to meet and terminate in the circumference at the point L, and join K L and L B, producing an inscribed right-angled parallelogram K L B H, to the circle. Produce H C to meet and terminate in L B, a side of the parallelogram K L B H, at the point N, and join N M. Bisect H N at V, and join V B. From the point O, the centre of the circle, draw a straight line through the point V, to meet and terminate in B M, the base of the right-angled triangle K B M, at the point T, and join T H. With B as centre and B F as interval, describe the arc F T.

Now, Sir, in this geometrical figure there are many things that will be — or, I should rather say, will appear to be — self-evident to any Mathematician. First: That B F = B T. Second: That the triangles K B M, K F H, and H P M are similar right-angled triangles. Third: Because K B is bisected at O, and B M at T; K B M and O B T are similar right-angled triangles. Fourth: Because

K B and L H are diagonals of the parallelogram K L B H ; K H B and B L K are similar right-angled triangles. Fifth : Because the diagonals of the parallelograms H N B M and H V B T intersect and bisect each other ; H B, the diagonal common to both parallelograms is bisected by O T, the hypothenuse of the right-angled triangle O B T. Sixth : H B is a diagonal of the right-angled parallelogram F B P H, and if we draw the other diagonal F P, these diagonals would also intersect and bisect each other at the point *n*, the point of intersection between the diagonals of the parallelograms H N B M and H V B T.

I shall not attempt to elaborate all the properties of this remarkable geometrical figure. I shall confine myself to demonstrating, by means of it, that *Euclid is at fault* in three of his most important Theorems. This I shall do in the simplest way possible, and so, bring my proofs within the capacity of anyone possessed of the most moderate geometrical and mathematical attainments.

Let K B, the diameter of the circle, = 8.
Then : by construction :

$$
\begin{aligned}
KH &= \tfrac{4}{5}(KB) = 6{\cdot}4\\
HB &= \tfrac{3}{4}(KH) = \tfrac{3}{5}(KB) = 4{\cdot}8.\\
BM &= \tfrac{3}{4}(KB) = \tfrac{3}{5}(KM) = 6.\\
HM &= \tfrac{3}{4}(HB) = 3{\cdot}6\\
KM &= KH + HM = 10.\\
BF &= BT = \tfrac{1}{2}(BM) = 3.\\
\text{And, } KF &= KB - BF = 5.
\end{aligned}
$$

Now, the triangles on each side of H B, are similar to the whole triangle K B M, and to each other ; and it follows of necessity, that $KB^2 - KH^2 = BM^2 - HM^2$; that is, the equation, $(8^2 - 6{\cdot}4^2) = (6^2 - 3{\cdot}6^2) = HB^2 = 23{\cdot}04$.

But, K H B is a right-angled triangle, and H F is perpendicular to K B, by construction ; therefore, according to Euclid, Prop. 8, Book 6, $KH^2 - KF^2$ should equal $HB^2 - BF^2$, and both should equal HF^2. But this is not a fact.

For :

$KH^2 - KF^2 = 6{\cdot}4^2 - 5^2 \quad 40{\cdot}96 - 25 = 15{\cdot}96$. But, $HB^2 - BF^2 = 4{\cdot}8^2 - 3^2 = 23{\cdot}04 - 9 = 14{\cdot}04$.

Therefore, it follows of necessity, that the triangles K F H and H F B are not similar to the whole triangle K H B, and to each other. Hence: the eighth proposition of the sixth book of Euclid is not of general and universal application, and consequently, *is not true*, under all circumstances. (Q. E. D.)

Again: Because H N is parallel to B M, B N parallel to H M, and H B a diagonal of the parallelogram H N B M, by construction; the triangles H B N and B H M are similar and equal right-angled triangles, and H B is the perpendicular of, and common to, both. Now, when K B the diameter of the circle = 8, then, H B = 4·8, B N = 3·6, and H N = 6. But, L B = K H, and K H = 6·4, when K B = 8; therefore, L B = 6·4. But, L B — N B = N L: therefore, 6·4 — 3·6 = 2·8 = N L.

Now, H B L is a right-angled triangle, and H N L a part of it, is an oblique-angled triangle; and according to Euclid: Prop. 12, Book 2, $\{HN^2 + NL^2 + 2(NL \cdot NB)\} = HL^2$; that is, $\{6^2 + 2{\cdot}8^2 + 2(2{\cdot}8 \times 3{\cdot}6)\} = (36 + 7{\cdot}84 + 20{\cdot}16) = 64 = HL^2$.

But, H F K is a right-angled triangle, and H O K a part of it, is an oblique-angled triangle, of which H O and O K sides of this triangle, are radii of the circle. But K F — O K = O F, by construction; and when K B the diameter of the circle = 8, H O and O K = 4, and O F = 1. Now $\{HO^2 + OK^2 + 2(OK \cdot OF)\} = \{4^2 + 4^2 + 2(4 \times 1)\} = (16 + 16 + 8) = 40$. But, when K B = 8, K H = 6·4; therefore, $6{\cdot}4^2 = 40{\cdot}96 = KH^2$ and is greater than 40; that is, greater than $\{HO^2 + OK^2 + 2(OK \cdot OF)\}$. Therefore, it follows of necessity, that the twelfth proposition of the second book of Euclid is not of general and universal application, and consequently *is not true*, under all circumstances. (Q. E. D.)

The 13th proposition of the second book of Euclid is simply the converse of the 12th; and it follows, that if the 12th proposition be fallacious, the 13th must necessarily be so, and there is no occasion to burden my Letter with the calculations, to prove it.

In the right-angled triangles K F H and H F B, H F is the perpendicular, and common to both. Now, to the Mathematician, it will appear, that the triangles K F H, H F B, and C B M, are similar right-angled triangles. But this is not so. For, if this were true, we should get the following analogy or proportion K B : B M : :

K F : F H, that is, 8 : 6 : : 5 : 3·75. But, $KH^2—KF^2$ is greater than 3·75, and therefore greater than F H ; and $HB^2—BF^2$ is less than 3·75, and therefore less than F H. Hence, K F H, H F B, and K B M are ***not*** similar triangles.

Where is the Mathematician who would have thought of doubting, that K F H and K B M are similar triangles? It is no doubt true that the angles in these triangles are similar, and one angle common to both ; but this does not make them similar triangles.

For : If K F H and K B M were similar right-angled triangles, we should get the analogy or proportion, K F : F B : : K H : H M. But, K F : F B : : K H : 3·84 ; this is, 5 : 3 : : 6·4 : 3·84, when K F + F B = 8. But, H M = 3·6 when the diameter of the circle = 8 ; therefore, K F H and K B M *cannot* be similar right-angled triangles.

Again : We should get the analogy or proportion, K B : B M : : H P : P M ; that is, 8 : 6 : : 3 : 2·25. But H P M is a right-angled triangle of which the sides H P and P M contain the right-angle ; therefore, $HP^2 + PM^2 = 3^2 + 2·25^2 = 9 + 5·0625 = 14·0625 = HM^2$; therefore, $\sqrt{14·0625} = 3·75 =$ H M. But, I have proved that H M = 3·6, when the diameter of the circle = 8 ; therefore, K B M and H P M cannot be similar right-angled triangles.

The sincere and earnest geometrical enquirer, will make many more remarkable discoveries, by means of this geometrical figure.

PROBLEM.

Construct a geometrical figure, in which there shall be two dissimilar and unequal right-angled triangles, of which the sides subtending the right angle shall be equal.*

This problem involves most important consequences to Mathematical science, and I question if there be a living Mathematician

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*In giving this problem, which I have stated very imperfectly, I had in my mind the geometrical figure represented in the diagram, enclosed in my Letter of the 9th November, 1868, to Mr. J. M. Wilson, (See Appendix C.) The reader will observe, by a reference to that Letter, how I should have stated the problem, so as to make my meaning distinctly intelligible. So far as "*Euclid at Fault*" is concerned, the problem might have been omitted.

competent to solve it. This, you, Sir, can readily find out, through the President of the Mathematical and Physical Section of the British Association. Had that Association permitted me to bring my discoveries before the Physical Section, they would have had the solution of this problem, and the consequences involved in it, before now.

It is my present determination never again to read a Paper or attempt to read a Paper, at the *British Association.* At the last Meeting, at Dundee, I offered, and was anxious, to read two Papers in the Mathematical and Physical Section; but the Committee of that Section obstinately persisted in their determination not to permit me. The subject of these Papers was Mean Proportionals, and in one of them my object was to shew that from any given determinate quantity (it may be commensurable or incommensurable), we may obtain two pairs of numbers, of which the mean proportional of both pairs, shall be the same finite quantity. This is a discovery in Mathematics, and leads to very important consequences. The *fact* is, the subject of mean proportionals may be reduced to a *theory*, and a most useful and valuable *theory* too, in Mathematical Science; and of this *fact* I believe our recognised *Mathematical Authorities* to be profoundly ignorant, at this moment. Is it not marvellous that, in an Association, called an Association for the Advancement of Science, the "*guiding stars*" of it should refuse to permit such subjects to be dealt with?

In conclusion: Strange as at first sight it may appear to you, Sir, for any one to assert that *Euclid is at fault;* if you should have read so far, you will have discovered that, "*'tis not more strange than true.*" In bringing this fact under your notice, *I have done my duty;* and it remains for you, as the forthcoming President of "*The British Association for the Advancement of Science,*" and as such, the guardian of its interests for a season, *to do yours.*

I am, Sir,
Yours very respectfully,
JAMES SMITH.

This was written on the 11th, and was in print in the form of a pamphlet by the 15th July, 1868. In my correspondence with the Rev. Professor Whitworth, I have admitted that there is a fallacy in the reasoning, and explained how I was led into it, by not observing that O B T and K B M are not similar right-angled triangles. The "*reasoning geometrical investigator*," however, who has read that correspondence, and carefully observed my explanation, will not fail to perceive how, by the geometrical figure represented by Diagram XIV., it may be proved that *all* the Theorems of Euclid ***are not of general and universal application.***

J. S.

FROM THE "ATHENÆUM" OF JULY 25TH, 1868.

OUR LIBRARY TABLE.

Euclid at Fault. By James Smith, Esq. (Simpkin, Marshall & Co.)

TURN about is fair play: we have often had "James Smith, Esq. at fault, by Euclid of Alexandria," and now we have the *converse.* This is a word on the meaning of which Mr. Smith requires instruction; he tells us that Euclid II. 13 is the converse of II. 12. We give Mr. Smith the present notice for two reasons. First, he has toned himself down since we last heard of him: he is so mild, that if he were but right no objection could be taken. Secondly, this Letter to the President of the British Association is the last communication that body is to receive from him. He offered to read them a paper "to show that from any given determinate quantity we may obtain two pairs of numbers, of which the mean proportional of both pairs shall be the same finite quantity." This they would not receive. But in announcing their punishment, Mr. Smith heightens it by pointing out something else which they have lost. "PROBLEM—Construct a geometrical figure, in which there shall be two dissimilar and unequal right-angled triangles, of which the sides subtending the right angle shall be equal. This problem involves most important con-

sequences to mathematical science, and I question if there be a living mathematician competent to solve it. . . . .
Had the Association permitted me to bring my discoveries before the Physical Section, they would have had the solution of this problem, and the consequences involved in it, before now." To mitigate the anguish of the Association, we will give the solution of the problem which days of deep thought have suggested to us: if the body be duly sensible, we shall not object to a testimonial. On the given hypothenuse A B describe a semicircle: take P and Q any points on the semicircle, so that A P and B Q are not equal; then A P B, A Q B, are the triangles required, right angled, unequal, dissimilar. Mr. James Smith professes to show that VI. 8 and II. 12, 13 are wrong. He does not show where Euclid's demonstrations fail, but he produces contradiction out of complicated constructions of his own. Now we can go with him thus far: we are satisfied that whenever he attempts to handle such constructions as he has shown, either he or Euclid will be at fault before he is out of the wood: accordingly, we advise him to keep himself in bottle nine years before he pronounces on his own flavour. We recommend him, not only to keep his word to the British Association, but to quit a field in which he has realized all the reputation of incompetency which it is given to man to obtain. The moment is favourable: here is Lord Napier of Magdala raised by the Queen to the House of Lords as the representative of calculated and organized success, almost at the moment when James Smith, Esq., of Liverpool, is nailed by himself to the barn-door as the delegate of miscalculated and disorganized failure.

APPENDIX B.

I commenced writing my printed Letter to His Grace the Duke of Buccleuch on the day it bears date, but it was not published for some weeks afterwards. I sent a copy to His Grace, so that it should fall into his hands before any one else could see it, and I find from the acknowledgment of its receipt, that I must have posted it on or about the 28th August, 1867. In the *Athenæum*, of September 7th, 1867, the receipt of it is acknowledged by the Editor in the list of new books, and he lost no time in handing it over to his Mathematical critic, Mr. A. de Morgan, as will be seen from the following critique:—

FROM THE "ATHENÆUM," SEPTEMBER 14TH, 1867.

OUR LIBRARY TABLE.

Letter to His Grace the Duke of Buccleuch...on the Quadrature and Rectification of the Circle. By James Smith, Esq. (Simpkin, Marshall & Co.)

Ecce iterum Crispinus, says the Latin poet; Here we are again, says the clown; May it please your Grace, says Mr. Smith. And it ought to do so; for the phrases are all fashioned upon the hypothesis that the Duke will read and understand, which is a compliment to his industry and intelligence that any Duke in Christendom might be proud of. His Grace will indeed be the "bold Buccleuch" if he does the first, and says the second. The circle is the old emblem of eternity; the symbol refers to the efforts to square it. The subdivisions of eternity are times. This time Mr. Smith brings on the stage the Rev. G. B. Gibbons, between whom and the $3\frac{1}{8}$-ist upwards of 120 letters have passed. We hope this means only six tens from one and half-a-dozen tens from the other; not 120 letters a-piece. Mr. Gibbons is the second mathematician whom Mr. James Smith has whiled into a long correspondence with him. Mr. De Morgan—who of course is handsomely acknowledged—showed his sense by never giving a private answer to any one of Mr. Smith's private letters. He knew that Mr. Smith is the Old Man of the Sea in the Arabian Nights, who would not dismount from the neck of any one who let him get up for a ride. Journals cannot be so served, and *we* are glad to see Mr. Smith again. We hope to have a bit of sport with him many a-time in the future, as we have had in the past. He is now an institution; and here we go round, round, round (and $\frac{1}{8}$ of a round, of course) is a regular part of our itinerary. As to the rest, there are two sines to an angle, geometrical and trigonometrical; one of them, no matter which, is greater than the measure of the angle in small angles; the mathematicians are a set of priests, who jealously guard a mystery; and Albert the Good will ever be revered, for when Mr. Smith sent one of his books, the Prince's librarian returned thanks for "the valuable addition made to His Royal Highness's Library.

JAMES SMITH *to the* EDITOR OF THE "ATHENÆUM."

BARKELEY HOUSE, SEAFORTH,
LIVERPOOL, 14*th September*, 1867.

SIR,

I beg to thank you for the complimentary notice of my Letter to his Grace the Duke of Buccleuch, on the Quadrature and Rectification of the Circle, which appears in the "*Athenæum*" of to-day. Complimentary it is, although you did not so intend it, and I ground this expression of opinion on the following facts. Assertion is not demonstration, neither is ridicule argument, and since you have not dared to point out any inadmissible step in my data, expose any fallacy in my reasonings, or attempt to grapple with any of my demonstrations by Logarithms, you have left my arguments unanswered, and my conclusions uncontradicted. Now, Sir, there is more in Geometry than was ever dreamed of in your mathematical philosophy, and I take the liberty of directing your attention to some geometrical and mathematical truths which will be perfectly new to you, and I shall make use of them to put your honesty and sincerity as a scientific Journalist to the test.

In the enclosed diagram *(See Diagram XV.)*, let the triangle A B C be the generating figure of the diagram, and represent a right-angled triangle of which the sides A B and B C which contain the right angle are in the ratio of 4 to 3, by construction. Then: With A as centre and A B as interval describe the circle X, and with B as centre and B A as interval describe the circle Y. Join the points D and E where the circumferences of the circles cut each other. Produce B A to meet and terminate in the circumference of the circle X at the point F, and join F D and F E, and so construct the equiangular and equilateral inscribed isosceles triangle F D E to the circle X. The lines D E and A B intersect and bisect each other at the point O ; and it follows, that the line F O which bisects the isosceles triangle F D E = 3 times O A, or O B = 2 B C; therefore, A O + $\frac{1}{2}$ O B = A G and B C. With A as centre and A G or B G as interval describe the circle Z. With A as centre and A C as interval, describe the circle X Y; and with A as centre and a side of the isosceles triangle, F D E as interval, describe the circle X Z. It is obvious that I might have drawn straight lines from the point A the centre of the circle X, parallel to F D and F E, and by joining the points where these lines would meet the circumference of the circle Y, have constructed an equilateral and equiangular inscribed isosceles triangle to the circle Y, two sides of which would be radii of the circle X Z.

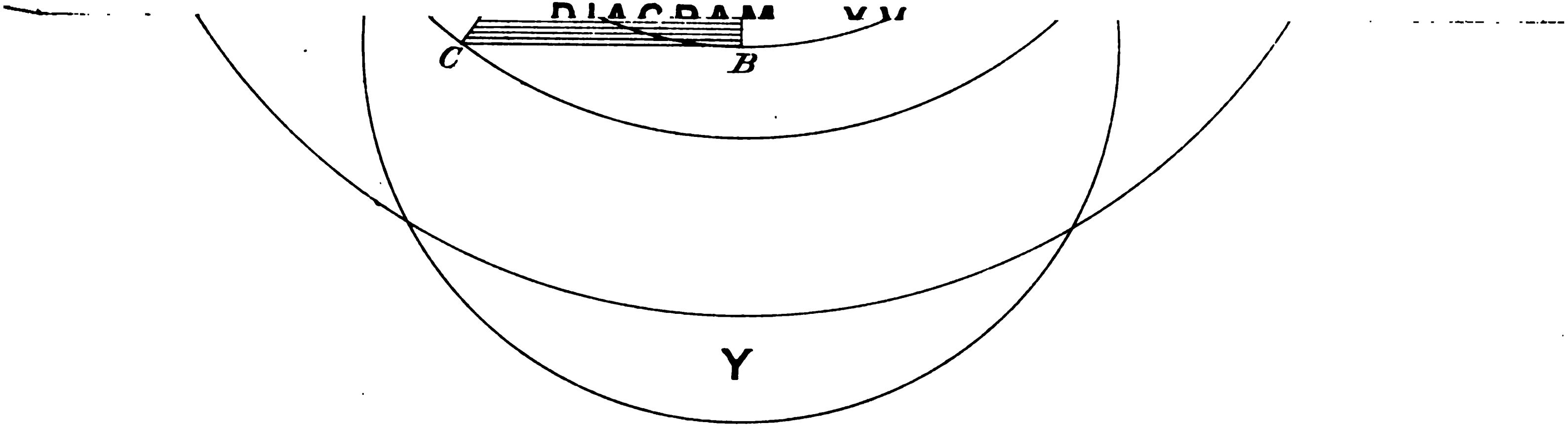
Now, let A B the radius of the circles X and Y = 4, which makes B C and A G = 3.

Then,

π (B C^2) or π (A G^2) = (3^2) = area of the circle Z.

π (A B^2) = π (4^2) = area of the circle X.

DIAGRAM XV



$\pi (AC^2) = \pi (5^2) =$ area of the circle X Y.

$\pi (DE^2) = \pi (\sqrt{48}^2) =$ area of the circle X Z.

3 times $AB^2 = DE^2$

$BC : AB :: DE^2 : FB^2$

$BC^2 : AB^2 ::$ area of circle Z : area of circle X.

And it follows, that if $BC : AB :: AB : n$, then, n times area of the circle Z = area of the circle X Z.

Hence :

B C is to A B as the area of a square on D E to the area of a circumscribing square to the circles X and Y. The area of the circle Z is to the area of the circle X or Y, as the area of a square on A G or B C to the area of a square on A B. The sum of the areas of the circles Z and X = area of the circle X Y. And, the area of the circle X Z is equal to $5\frac{1}{3}$ times the area of the circle Z: 3 times the area of the circle X: and 1·92 times the area of the circle X Y.

Let the letters which represent the circles denote the arithmetical values of their areas. Then :

$XZ : 5\frac{1}{3} :: Z :$ unity.

$XZ : 3 :: X :$ unity.

$XZ : 1{\cdot}92 :: XY :$ unity.

All the foregoing facts are quite independent of the true arithmetical value of π. In other words, for the purpose of ascertaining the relative values of the circles, we may hypothetically adopt 3·1416, 3·14159, or, 3·14159265 as the value of π, or we may work out the calculations on any hypothetical value of π greater than 3 and less than 4, and so, vary the arithmetical values of the areas of the circles, but we cannot alter the ratios or relative values of circle to circle ; that is to say, $XZ : 5\frac{1}{3} :: Z :$ unity ; by whatever hypothetical value of π we may ascertain the values of X Z and Z.

Again : 3 (X Z), 16 (Z), and $4\pi (FO^2)$, are equivalent expressions, and if these expressions be denoted by P, then P = area of a circle of which F O the bisecting line of the isosceles triangle F D E is the semi-radius. One hypothetical value of π intermediate between 3 and 4 is just as good as another for the purpose of demonstrating these facts.

Now, the following *fact* is inseparably connected with the geometrical figure represented by the diagram :—

$(AB^2 + BC^2 + AC^2) = 3{\cdot}125\ (AB^2)$ whatever arithmetical value we may put upon A B the radius of the circles X and Y; that is to say, the sum of the squares of the three sides of the right-angled triangle A B C the generating figure of the diagram = 3·125 times the area of a square on A B the perpendicular.

Now, $\frac{\text{Area of the circle Z}}{FO^2} = \frac{\pi}{4}$; and, $\frac{\pi}{4}$ = area of a circle of diameter unity, whatever be the value of π. That the expression $\frac{\text{Area of the circle Z}}{FO^2} = \frac{\pi}{4}$, may be demonstrated by means of any hypothetical value of π intermediate between 3 and 4. For example: by hypothesis, let $\pi = 3{\cdot}14159265$, and let A B the radius of the circles X and Y = 4. Then: B C the base of the right-angled triangle $ABC = \frac{3}{4}(AB) = 3$, by construction. But, A G = B C, by construction, and A G is the radius of the circle Z; therefore, $\pi\ (AG^2$ or $\pi\ (BC^2) = 3{\cdot}14159265 \times 9 = 28{\cdot}27433385$ = area of the circle Z. $\frac{\text{Area of the circle Z}}{FO^2} = \frac{28{\cdot}27433385}{36} = \frac{3{\cdot}14159265}{4} = {\cdot}7853981625 = \frac{\pi}{4}$. Again: By hypothesis, let $\pi = 3{\cdot}1416$, and let F B the diameter of the circle X = unity = 1. Then: A B the radius of the circles X and $Y = \frac{1}{2} = {\cdot}5$; and A G or $BC = \frac{3}{4}(AB) = {\cdot}375$. Therefore, $\pi(AG^2)$ or $\pi(BC^2) = 3{\cdot}1416 \times {\cdot}375^2 = 3{\cdot}1416 \times {\cdot}140625 = {\cdot}4417875$ = area of the circle Z. $\frac{\text{Area of the circle Z}}{FO^2} = \frac{{\cdot}4417875}{{\cdot}75^2} = \frac{{\cdot}4417875}{{\cdot}5625} = {\cdot}7854 = \frac{\pi}{4}$. Now, my good Sir, any other hypothetical value of π intermediate between 3 and 4 will produce similar results, and if you are incompetent to convince yourself of this fact, you must be more of a "*clown*" than a philosopher.

Well, then, on the THEORY that 8 circumferences of a circle are exactly equal to 25 diameters, which makes $\frac{25}{8} = 3{\cdot}125$ the true arithmetical value of π, $(AB^2 + BC^2 + AC^2) = \frac{\text{Area of the circle Z}}{FO^2} = \frac{\pi}{4}$ = area of the circle X, when F B the diameter of the circle X = unity = 1.

Proof: If F B = unity = 1, A B the radius of the circles X and $Y = \frac{1}{2} = {\cdot}5$; and, $\frac{3}{4}(AB) = {\cdot}375$ = B C the base of the triangle A B C, and B is a right angle, by construction; therefore, AB^2 +

$BC^2 = {\cdot}5^2 + {\cdot}375^2 = {\cdot}25 + {\cdot}140625 = {\cdot}390625 = AC^2$; therefore, $AB^2 + BC^2 + AC^2 = {\cdot}25 + {\cdot}140625 + {\cdot}390625 = {\cdot}78125$ = the sum of the areas of squares on the sides of the right-angled triangle A B C, the generating figure of the diagram, and is equal to $\frac{\pi}{4}$ = area of a circle of diameter unity. But, $\pi\,(BC^2) = 3{\cdot}125 \times {\cdot}375^2 = 3{\cdot}125 \times {\cdot}140625 = {\cdot}439453125$ = area of the circle Z; and F O the bisecting line of the isosceles triangle F D E = $\frac{3}{4}$ (F B) = $\frac{3}{4}$ (unity) = ·75; therefore, $\frac{\text{Area of the circle Z}}{FO^2} = \frac{{\cdot}439453125}{{\cdot}75^2} = \frac{{\cdot}439453125}{{\cdot}5625} = {\cdot}78125 = \frac{\pi}{4}$ = area of a circle of diameter unity. Hence: We get the equation $(AB^2 + BC^2 + AC^2) = \frac{\text{Area of the circle Z}}{FO^2}$, and this equation $= \frac{\pi}{4}$, whatever arithmetical value we may be pleased to put upon A B, the radius of the circles X and Y.

Now, Sir, if you can find any other value of π either greater or less than 3·125, by which you can obtain this equation, you will be something more than a philosopher, and will prove to the scientific world that you are indeed a phenomenon, even more remarkable than "*the Old Man of the Sea, in the Arabian Nights.*"

Again, Sir, you will admit that geometrical data admit of no doubt, and I shall now adopt a *datum* with regard to which it is impossible you can raise an objection. In every circle, the perimeter of a regular inscribed hexagon = 6 times radius, and $\frac{3}{\pi}$ expresses the ratio between the perimeter of every regular hexagon and the circumference of its circumscribing circle, whatever be the value of π.

Well, then, if the radius of a circle = 1, the perimeter of a regular inscribed hexagon = 6. Let this be our *datum*, and if upon this datum I stumble into illogical reasoning, it will be for you to prove it, and so vitiate my conclusions.

Now, $\frac{3}{3{\cdot}125}(6) = 5{\cdot}76$, and $\frac{3{\cdot}125}{3}(6) = 6{\cdot}25$; therefore, $5{\cdot}76 : 6 :: 6 : 6{\cdot}25$, and the mean proportional between 5·76 and 6·25 is $\sqrt{5{\cdot}76 \times 6{\cdot}25} = \sqrt{36} = 6$ = the perimeter of a regular inscribed hexagon to a circle of radius 1. Hence: $\frac{3}{3{\cdot}125}$ expresses the ratio between the perimeter of every regular hexagon and the circumference of its circumscribing circle.

Proof: The following is an example of continued proportion: $a : b :: b : c :: c : d :: d : e.$

Now:

$\frac{3}{3{\cdot}125}(e) = d$: $\frac{3}{3{\cdot}125}(d) = c$: $\frac{3}{3{\cdot}125}(c) = b$: and $\frac{3}{3{\cdot}125}(b) = a$. And, when $e = 6{\cdot}25$, then, $d = 6$: $c = 5{\cdot}76$: $b = 5{\cdot}5296$: and $a = 5{\cdot}308416$: therefore, $\frac{3{\cdot}125}{3}(a) = b$: $\frac{3{\cdot}125}{3}(b) = c$: $\frac{3{\cdot}125}{3}(c) = d$: and, $\frac{3{\cdot}125}{3}(d) = e$.

Hence:

$b \times d = a \times e$, and c the middle term is a mean proportional between a and e the extreme terms, and is also a mean proportional between b and d. b is a mean proportional between a and c, and d is a mean proportional between c and e.

Therefore:

When e = circumference of a circle, d = perimeter of a regular inscribed hexagon.

When d = circumference of a circle, c = perimeter of a regular inscribed hexagon.

When c = circumference of a circle, b = perimeter of a regular inscribed hexagon.

When b = circumference of a circle, a = perimeter of a regular inscribed hexagon.

Now, Sir, we may put any arithmetical value we please upon e, and in working out the calculations backwards from e to a obtain exact arithmetical results; but if we put an arithmetical value upon a and attempt to work out the calculations forwards from a to e (unless we first get the value of a by making the calculations backwards from e to a) we get into incommensurables immediately. For example: If $a = 6{\cdot}25$, $\frac{3{\cdot}125}{3}(a) = \frac{3{\cdot}125 \times 6{\cdot}25}{3} = \frac{20{\cdot}3125}{3} = 6{\cdot}770833$ with 3 to infinity. But, will you dare to tell me that for this reason, the middle term c is not, and cannot be, a mean proportional between the extreme terms a and e? You would indeed be more of a mathematical "*clown*" than a philosopher if unable to perceive that the difficulty (if difficulty it can be called) arises from the mere fact that all numbers are not divisible by 3 and its multiples without a remainder.

Now, Sir, you will not venture to dispute that $\frac{3}{\pi}$ and $\frac{6}{2\pi}$ are equivalent ratios, and that both express the ratio between the perimeter of every regular hexagon and the circumference of its circumscribing circle, whatever be the value of π ; and I must now direct your attention to the following formula :—

$\frac{2\pi}{6}(a) = b$, $\frac{2\pi}{6}(b) = c$, and $\frac{2\pi}{6}(c) = d$, and so on, *ad infinitum.*

Hence: If a represents the perimeter of a regular hexagon, b represents the circumference of its circumscribing circle. If b represents the perimeter of a regular hexagon, c represents the circumference of its circumscribing circle. If c represents the perimeter of a regular hexagon, d represents the circumference of its circumscribing circle, and so on, *ad infinitum.*

Will you dare to dispute the formula because you cannot put a value on a and work out the calculations from a to d with arithmetical exactness? I trow not! Well, then, to vitiate my conclusion that 8 circumferences of a circle are exactly equal to 25 diameters, which makes $\frac{25}{8} = 3{\cdot}125$ the true arithmetical value of π. you must not only dispute my data, and prove my reasoning to be illogical, but you must find an arithmetical expression for the ratio between the perimeter of a regular hexagon and its circumscribing circle, by which you can put a value on d, work out the calculations backwards to a, and prove that b is a mean proportional between a and c, and c a mean proportional between b and d. Think of "*The Old Man of the Sea, in the Arabian Nights.*"

Again : In the triangle A B C, the generating figure of the diagram, the sides A B and A C are in the ratio of 4 to 5, by construction.

Now, A B : A C : : A C : m : : m : n : : n : p is an example of continued proportion, and when A B + $\frac{1}{4}$ (A B) = A C, then, A C + $\frac{1}{4}$ (A C) = m: $m + \frac{1}{4}(m) = n$: and $n + \frac{1}{4}(n) = p$; and it follows, that, when A B = 4, A C = 5: $m = 6{\cdot}25 = 2\pi$: $n = 7{\cdot}8125$ = 10 times the area of a circle of diameter unity: and $p = \pi^2 = 3{\cdot}125^2 = 9{\cdot}765625$.

Hence:

A B × n = A C × m = $10(\pi) = 31{\cdot}25$.
A C × p = $m \times n$ = $5(\pi^2) = 48{\cdot}828125$.
A B × p = A C × n = $(2\pi)^2 = m^2 = 39{\cdot}0625$.

Therefore:

The middle term m is a mean proportional between the extreme terms A B and p, and is also a mean proportional between the terms A C and n. A C is a mean proportional between A B and m, and n is a mean proportional between m and p.

Now, let x denote the area of a square, and let y denote the area of its circumscribing circle.

Then:

$\{(x + \frac{1}{4}x) + \frac{1}{4}(x + \frac{1}{4}x),\}$ and $2x\left(\frac{\pi}{4}\right)$, are equivalent expressions, and both equal to y.

Thus, from the foregoing facts we obtain a method of finding the area of a circle from the given area of its inscribed square, which shivers to atoms the absurd notion of Mathematicians that π can only be represented arithmetically by an infinite series.

Now, Sir, you have charged me with being a fool, or at any rate you have endorsed the charge by permitting Mr. de Morgan so to call me repeatedly in the columns of your Journal. You profess to be a Mathematician, and if so, you will point out where there is a fallacy in my reasoning, and thus vitiate my conclusions, establish Mr. de Morgan's charge, and put me to silence. If you find this to be impossible, and are an honest scientific Journalist, you will purge your conscience by admitting my conclusions through the columns of the *Athenæum*, and so far as you are concerned, acquit yourself of any participation in Mr. de Morgan's scurrility and ribald vulgarity.

I shall now assume a thing which remains to be proved. In this communication, "*the phrases are all fashioned upon the* HYPOTHESIS" that you, Mr. Editor, are a philosopher and a gentleman, and consequently, that "*you will read and understand, which is a compliment to your industry and intelligence.*" Now, if my hypothesis be well founded, you will mark and inwardly digest this epistle, and doing so, you cannot fail to be thereby translated from the kingdom of mathematical darkness, into the kingdom of geometrical light, and thus become a "3⅛-*ist;*" and you will prove to the scientific world that you are an honest convert to geometrical truth by making a frank confession of your faith in 3⅛-ism, through the columns of the leading scientific Journal.

Now, let the area of a square be represented by any arithmetical quantity, and be given to find two pairs of numbers, of which the mean proportional between both pairs of numbers shall be equal to the given area of the square. The larger numbers in one pair not to be multiples or submultiples of the larger numbers in the other. For example: Let the given area of the square be 7. Then: 14 and 3·5 will be the first pair of numbers, and 10·9375 and 4·48 will be the second pair of numbers, and $\sqrt{14 \times 3{\cdot}5} = \sqrt{10{\cdot}9375 \times 4{\cdot}48}$, and both expressions $= \sqrt{49} = 7$; therefore, the given area of the square is a mean proportional between either pair of numbers.

Again: Let the sides of a square be represented by any arithmetical quantity, and be given to find two pairs of numbers of which the mean proportional between both pairs of numbers shall be represented by the same arithmetical symbols. As in the former example, the larger numbers in one pair not to be multiples or submultiples of the larger numbers in the other. For example: Let the given side of the square be $\sqrt{5}$. In this case we may work out the calculations either outwards or inwards. If outwards, the first pair of numbers will be 10 and 2·5, and the second pair of numbers will be 7·8125 and 3·2, and $\sqrt{10 \times 2{\cdot}5} = \sqrt{7{\cdot}8125 \times 3{\cdot}2}$, and both expressions $= \sqrt{25} = 5$; therefore, the mean proportional between either pair of numbers is represented by the same arithmetical symbol. If we work inwards, the first pair of numbers will be 5 and 1·25, and the second pair of numbers will be 3·90625 and 1·6, and $\sqrt{5 \times 1{\cdot}25} = \sqrt{3{\cdot}90625 \times 1{\cdot}6}$, and both expressions $= \sqrt{6{\cdot}25} = 2{\cdot}5$; therefore, the mean proportional between either pair of numbers is represented by the same arithmetical symbols.

Now, Sir, permit me to put the following questions:—Do you know the rule by which these peculiar mean proportionals are found? If you are a philosopher and a gentleman, will you not answer this question by giving the rule? If you do not know the rule and are a gentlemen, will you not admit the fact? If you decline to do either, and fancy you are protected by the privileges connected with your editorial capacity, will you not falsify my *hypothesis*, and prove that you are neither a philosopher nor a gentleman? In putting these questions do I not fulfil the promise made in the first sheet of this communication, and put your honesty and sincerity as a scientific Journalist to the test? You say in your notice of my Letter to the Duke of Buccleuch, that "*you hope to have a bit of sport with me many a time in the future, as you have had in the past.*" Now's your time. This epistle affords you the opportunity. Don't fail to let me share in the "*sport*" you get out of it.

In conclusion: You charge me with having "*whiled*" two Mathematicians into a long correspondence. This is not true! I "*whiled*" neither of the gentlemen to whom you refer into any correspondence. The correspondence was of their seeking, not mine, and both are personally unknown to me at this moment. Do not imagine, my good Sir, that I have any desire to play the part of a WILY disputant and attempt to *while* you into a long correspondence. I have no such inclination, I can assure you. But, I confess I should like YOU to look upon this communication as my last effort to *wile* or *wheedle* you and Professor de Morgan into honest Mathematicians, and if the pair of you have "*the power to see truth, and the candour to admit it,*" I shall most assuredly be successful, and no further correspondence on my part will be necessary, as to the ratio of diameter to circumference in a circle.

I am, Sir,
Yours respectfully,
JAMES SMITH.

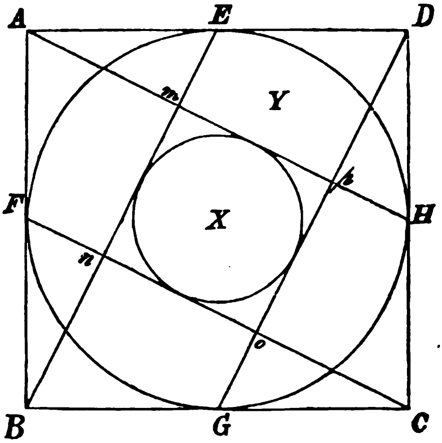
JAMES SMITH *to* THE EDITOR OF THE AEHENÆUM.

BARKELEY HOUSE, SEAFORTH,
21st September, 1867.

SIR,

In your publication of to-day, I observe, in "Notices to Correspondents," your acknowledgment of the receipt of my Letter of the 14th inst.

Now, Sir, I recognize, and admit, the distinction between you as the Editor of, and Professor de Morgan as the mathematical critic to, the leading scientific Journal; and *I know* that you, Mr. Editor, are sufficient of a Mathematician to "*read and understand*" the facts I am about to bring under your notice; and the question I shall put to you, arising out of these facts, you are competent to answer, without the assistance of either Professor de Morgan, or any other of our great "*mathematical authorities.*"



In this geometrical figure, let A B C D represent a square. Bisect the sides at the points E F G, and H. Join E B, D G, A H, and F C. In the square *m n o p* inscribe the circle X, and in the square A B C D inscribe the circle Y. Then: The square *m n o p* is equal to one fifth part of the square A B C D; and it follows of necessity, that the area of the circle X inscribed in the square *m n o p*, is equal to one-fifth part of the area of the circle Y inscribed in the square A B C D.

Now, let the area of the circle X be represented by any finite arithmetical quantity, and be given to find two pairs of numbers, so that the smaller number in either pair shall be an aliquot part, or submultiple of the larger number in the other pair, and the mean proportional between both pairs of numbers of the same arithmetical value.

Well, then, let the area of the circle X be represented by 666, the number of the Apocalyptic beast. Then: One pair of numbers will be 3330 and 832·5, and the other pair of numbers will be 2601·5625 and 1065·6, and $\sqrt{3330 \times 832{\cdot}5} = \sqrt{2601{\cdot}5625 \times 1065{\cdot}6} = \sqrt{2771225} = 1665$; that is, the mean proportional between 3330 and 832·5 is represented by the same arithmetical symbols as the mean proportional between 2601·5625 and 1065·6. Hence: $\frac{3330}{1065{\cdot}6} = \frac{2601{\cdot}5625}{832{\cdot}5}$, and this equation or identity = 3·125; therefore, $\frac{3330}{1065\cdot6}$ and $\frac{2601\cdot5625}{832\cdot5}$ are equivalent ratios, and both equivalent to the ratio $\frac{3\cdot125}{1}$. Again: $\frac{3330}{832\cdot5} = 4$, and $\frac{2601\cdot5625}{1065\cdot6} = \left(\frac{3\cdot125}{2}\right)^2 = 1\cdot5625^2 = 2\cdot44140625$. Hence: The mean proportional between 4 and 2·44140625 = $\sqrt{4 \times 2\cdot44140625} = \sqrt{9\cdot765625} = 3\cdot125$. But further: $\frac{3330}{2601\cdot5625}$ and $\frac{1065\cdot6}{832\cdot5}$ are equivalent ratios; and both equivalent to the ratio $\frac{1\cdot28}{1}$; and 1·28 : 1 :: 1 : ·78125; therefore, $\frac{1\cdot28}{1}$ and $\frac{1}{\cdot78125}$ are equivalent ratios, and both express the ratio between the area of every square and the area of its inscribed circle.

Again: Let the area of the circle X be represented by the digit 5, which stands in the centre of our system of arithmetical notation, and is the arithmetical mean between the extremes of 1 and 9, 2 and 8, 3 and 7, and 4 and 6. Then: One pair of numbers will be 25 and 6·25; and the other pair will be 19·53125 and 8; and $\sqrt{25 \times 6\cdot25} = \sqrt{19\cdot53125 \times 8} = \sqrt{156\cdot25} = 12\cdot5$. Hence: The mean proportional between 25 and 6·25 is represented by the same arithmetical symbols as the mean proportional between 19·53125 and 8, and is equal to 4 π. Hence: $\frac{25}{8}$ and $\frac{19\cdot53125}{6\cdot25}$ are equivalent ratios, and both equivalent to the ratio $\frac{3\cdot125}{1}$. Again: $\frac{25}{6\cdot25} =$ 4, and $\frac{19\cdot53125}{8} = \left(\frac{3\cdot125}{2}\right)^2 = 1\cdot5625^2 = 2\cdot44140625$. Hence: The mean proportional between 4 and 2·44140625 = $\sqrt{4 \times 2\cdot44140625} = \sqrt{9\cdot765625} = 3\cdot125 = \pi$, as in the previous example. But further: $\frac{25}{19\cdot53125}$ and $\frac{8}{6.25}$ are equivalent ratios, and both equivalent to the ratio $\frac{1\cdot28}{1}$, and 1·28 : 1 :: 1 : ·78125; that is, 1·28 : 1 :: 1 : $\frac{\pi}{4}$; therefore, $\frac{1\cdot28}{1}$ and $\frac{1}{\cdot78125}$ are equivalent ratios, and both express the ratio between the area of a square and the area of its inscribed circle.

Now, Sir, there is no mathematical magic in finding two pairs of numbers from a given area of the circle X, of which the mean proportional between the two pairs of numbers shall be of the same arithmetical value. All that is required is to know the rule, and I put the following plain question to you :—Do you know the rule ? If you know it, will you not, as an honest scientific journalist, let your readers have the benefit of your knowledge ? If you know it not, pray have the candour to admit it ! *I know* the rule, and have no wish to keep it a secret, and shall have much pleasure in revealing the secret to you, on receiving your assurance, that you will give currency to it through the columns of the *Athenæum*.

I am, Sir,
Yours respectfully,
JAMES SMITH.

FROM THE "ATHENÆUM," SEPTEMBER 28TH, 1867.

OUR WEEKLY GOSSIP.

"Bless the man !" said Miss Trotwood, of Micawber, "he would write letters by the ream if it were a capital offence." Mr. James Smith would do as much even were it a reasonable thing ; so fond is he of writing. Fifteen quarto pages, mostly of geometry, in answer to our "complimentary" notice of his Letter to the Duke of Buccleuch. Complimentary he calls it, because we have left his arguments unanswered: surely he does not mean that this is the first compliment we have paid him ? And ridicule is not argument: we know that; if ridicule had been argument we should never have ridiculed Mr. James Smith. He calls us to repentance and to $3\frac{1}{8}$; he says his Letter is written to test our sincerity. How he does forget! He found out that we were impostors long ago. In one point he has hit us ; we wrote *while* instead of *wile :* we remember how it was ; we began to write *wheedle*, and changed it into *w(h)ile* in the act of writing. And so we are to *argue* with a man who produces a pentagon of which four sides are together *geometri-*

cally larger than the fifth, but *mathematically* equal to it. Surely, the *argument* of the Carol, which appeared in the *Correspondent* is good enough. The first verse is as follows—

A creed is a very fine thing,
Above all when there is'nt much of it;
There is no π but three and one-eighth,
And Mr. James Smith is its prophet.
So here we go round, round, round,
And there we go square, square, square,
Five per cent. is a bob in the pound,
And sixpence an omnibus fare.

We have been told on good authority that the last line of the chorus was at first—

And J—— S—— is a d——y who's there.

Really Mr. J. S. deserves the restoration of the original reading, painful as it is to us to make it known.

In the *Athenæum*, of January 4th, 1868: there appeared an Article bearing the signature of A. de Morgan, headed:—"*Pseudomath, Philomath, Graphomath.*" (*See Appendix D*). The Article commences thus:—"*Many thanks for the present of Mr. James Smith's Letters of September 28th, and October 10th and 12th.*" Many thanks to whom, if not to the *recognised* Editor of the *Athenæum*? Not a word from the beginning to the end of the Article, about Mr. James Smith's Letters of September 14th and 21st, 1867. Does Mr. A. de Morgan mean to say, that these Letters were not presented to him by the Editor of the *Athenæum*? Does he not convey the impression to the readers of that Journal, and intend to convey the impression, that he was not the writer of the *scurrilous and* untruthful critique upon these Letters which appeared in the *Athenæum* of September 28th, 1867? Does he not assume, *a* *la Talleyrand*, that "*language was given to man, not to reveal, but to conceal his thoughts?*" These are not only sound inferences, but the only inferences, that can be drawn from Mr. A. de Morgan's omission of all reference to Mr. James Smith's Letters of September 14th and 21st, 1867. I do not think a more barefaced piece of *chicanery* was ever perpetrated by a public writer! If Mr. A. de Morgan was not the writer of the Article in question, who was? By implication, he makes the "*King of English Critics*" the writer. I must leave the "*King of English Critics*," and, the *King of English Sophists* to settle this point between themselves. Well, then, Mr. A. de Morgan admits the fact, that Mr. James Smith's Letters of September 28th, and October 10th and 12th, were presented to him by the Editor of the *Athenæum.*

JAMES SMITH *to* THE EDITOR OF THE "ATHENÆUM."

BARKELEY HOUSE, SEAFORTH,
28th September, 1867.

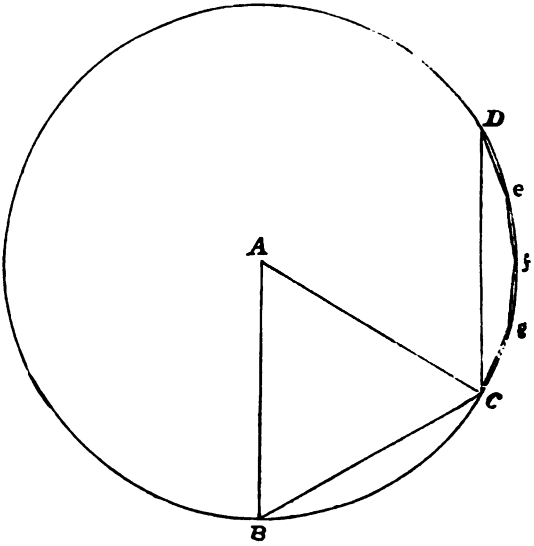
SIR,

Since the "*complimentary*" notice of my Letter to his Grace the Duke of Buccleuch on the Quadrature and Rectification of the Circle, which appeared in the *Athenæum* of the 14th inst., I have addressed two Letters to you, Mr. Editor, written under the *impression* that they would fall into the hands of Mr. William Hepworth Dixon. If my *impression* be opposed to fact, you, Mr. Editor—be you who you may—can prove it; and unless you furnish the proof, I shall continue to assume that Mr. William Hepworth Dixon is the *recognised* Editor of the leading scientific Journal; and if not the writer, at any rate the supervisor, and approver, of all that appears in "*Our Library Table*" and "*Our Weekly Gossip.*"

Now, all the phrases in my two Letters were "*fashioned upon the hypothesis*," that *you*, Mr. William Hepworth Dixon, are a philosophical Editor; and consequently, that you can "*read and understand*" a communication on a scientific subject. Am I to conclude that this *hypothesis* is not founded upon fact? Am I to infer that you are not a philosophical Editor, and consequently, that you cannot *understand* a scientific communication? If my *hypothesis* be well founded, how was it that my Letters were handed over to Professor de Morgan to be dealt with after his *scurrilous* fashion? How happens it that you (Mr. William Hepworth Dixon) appear to merely play the part of a puppet, to a mathematical "*impostor*," who assumes that he has "*the capacity to criticise a work before it is either published or written*," *and are ready to insert anything, at his bidding, in the leading scientific Journal, no matter how vulgar, contemptible, and untruthful? Witness the ***untruthful and absurd trash*** that appears in "*Our Weekly Gossip*" in this day's ***Athenæum.***

"*Here we are, says the clown.*" "***And so we*** (how wondrous is that little word the editorial *we*) ***are*** "***to argue with a man who produced a pentagon of which four sides are together geometrically larger than the fifth, but mathematically equal to it.***"

FIG. 1.



In this geometrical figure (Fig. 1) let AB be a straight line of any length. With A as centre and AB as interval, describe the circle, and on A B describe the equilateral triangle ABC. From the point C draw a straight line parallel to AB, to meet and terminate in the circumference of the circle at the point D. Then: D C = A B. Divide the arc subtending the chord D C into four equal parts, and draw the chords D *e*, *ef*, *fg*, and *g* C.

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* See my pamphlet, "*The British Association in Jeopardy, and Professor de Morgan in the pillory, without hope of escape.*"

Now, D C = A B = radius of the circle, by construction ; and 2π (radius) = circumference in every circle ; therefore, 2π (D C) = circumference of the circle, and it follows, that $\frac{\pi}{3}$ (D C) = the arc D *f* C ; that is, $\frac{\pi}{3}$ (DC) = an arc subtending a chord equal to radius, and is therefore equal to one-sixth part of the circumference of the circle ; therefore, $\frac{\frac{\pi}{3}(\mathrm{D\,C})}{4}$ = one twenty-fourth part of the circumference of the circle.

Hence: if the circumference of the circle be divided into any number of equal arcs, $\frac{3}{\pi}$ (arc) multiplied by the number of arcs = the perimeter of a regular inscribed hexagon to the circle. For example: By hypothesis, let $\pi = 3{\cdot}1416$ and D C = ·5. Then: 2π (D C) = $6{\cdot}2832 \times {\cdot}5 = 3{\cdot}1416$ = circumference of the circle. Now, let the circumference of the circle be divided into 96 equal arcs. Then: $\frac{3{\cdot}1416}{96} = {\cdot}032725$ = one of these arcs. $\frac{3}{\pi}$ (arc) $= \frac{3 \times {\cdot}032725}{3{\cdot}1416} = {\cdot}03125$; therefore, 96 times ·03125 = 3 = the perimeter of a regular inscribed hexagon to the circle. Again: By hypothesis, let $\pi = 3{\cdot}1416$, and D C = 16. Then: 2π (D C) = $6{\cdot}2832 \times 16 = 100{\cdot}5312$ = circumference of the circle. $\frac{100{\cdot}5312}{24} = 4{\cdot}1888$ = each of the arcs subtending the chords D *e*, *e f*, *f g*, and *g C*; therefore, 24 times $\frac{3}{\pi}(4{\cdot}1888) = 24\left(\frac{3 \times 4{\cdot}1888}{3{\cdot}1416}\right) = 24\left(\frac{12{\cdot}5664}{3{\cdot}1416}\right) = 24 \times 4 = 6$ (D C) $= 6 \times 16 = 96$ = the perimeter of a regular inscribed hexagon to the circle ; therefore, 24 times $\frac{\pi}{3}(4) = 24\left(\frac{3{\cdot}1416 \times 4}{3}\right) = 24\left(\frac{12{\cdot}5664}{3}\right) = 24 \times 4{\cdot}1888 = 100{\cdot}5312$ = circumference of the circle. Now, let the circumference of the circle be divided into 96 equal arcs. Then: $\frac{100{\cdot}5312}{96} = 1{\cdot}0472$

= one of these arcs, and $\frac{3}{\pi}$ (arc) $= \frac{3 \times 1\cdot0472}{3\cdot1416} = 1$; therefore, 96 times $1 = 6$ (D C) $= 96 =$ the perimeter of a regular inscribed hexagon to the circle, and $6\ (1\cdot0472) = 6\cdot2832 = 2\ \pi$. Hence : $\frac{\pi}{3}(4) = \frac{3\cdot1416 \times 4}{3} = \frac{12\cdot5664}{3} = 4\cdot1888$ is the value of the arcs subtending the chords D *e*, *ef*, *fg*, *and g C*, on the hypothesis that $\pi = 3\cdot1416$.

Now, Sir, you and your coadjutor, Professor de Morgan, may cloak these facts in any language you like ; and you may garble and pervert my language and meaning in any way you please ; but you will never either "*w(h)ile*" or "*wheedle*" any honest man, of moderate mathematical attainments, into the belief, that when the radius of a circle $= 16, \frac{\pi}{3}(4)$ is not equal to one twenty-fourth part of the circumference, or that $6\left(\frac{\text{circumference}}{96}\right)$ is not equal to $2\ \pi$, whatever be the value of π.

Now, I have before me a modern edition of "*Euclid's Elements of Plane Geometry, with explanatory appendix, adapted* for *the use of schools, or for self-instruction.*" The compiler, in his "*explanatory appendix* to the fifth and sixth books, observes:—" *The same proposition (Prop* 13, *Book* 6*) which enables us to find a mean proportional between two given lines, will also enable us to find a mean proportional between the first and second, and between the second and third: and thus to interpolate mean proportionals between the terms to any extent. But to find two mean proportionals, or,* A *and* B *being given, to find x and y, so that* A $: x :: y :$ B, *is a problem beyond the reach of Plane Geometry.*" If the compiler meant to say, that by the aid of Mathematics we cannot find two mean proportionals, so that A $: x :: y :$ B, or, in other words, that we cannot by the aid of Mathematics find two mean proportionals, so that the mean proportional between A and B the *extremes*, shall be equal to the mean proportional between x and y the *means*, his assertion is not true, and this I shall now proceed to prove.

The construction of this geometrical figure (Fig. 2) may be described as follows ; On A B, a line of any length, describe the square A B C D.

Draw the diagonal A C, and on A C describe the square A C E F.

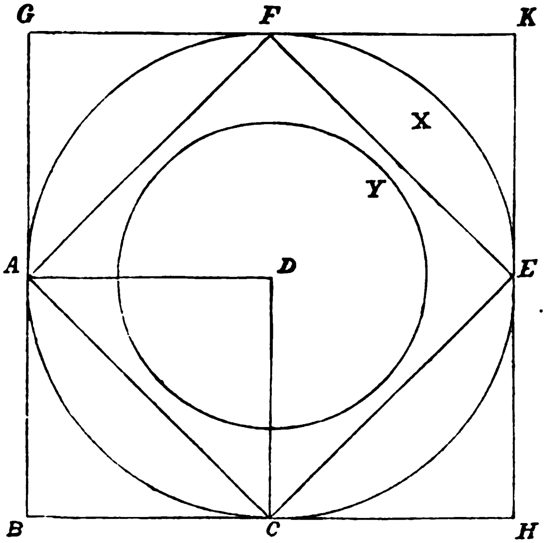

FIG. 2.

With D as centre and D A or D C as interval, describe the circle X, and about it circumscribe the square G B H K. With D as centre, and any interval greater than half the side of the square A B C D, but less then half the side of the square A C E F, describe the circle Y, the radius of which shall be for the present unknown.

Let A denote the area of the square G B H K circumscribed about the circle X. Let B denote the area of the square A B C D, on the radius of the circle X. Let C denote the area of the square on A C, a diagonal of the square A B C D, and therefore an inscribed square to the circle X.

Then : $\frac{1}{4}$ (A) = B, and the mean proportional between A and B = C, that is, $\sqrt{A \times B} = C$, whatever arithmetical value we may put upon A. Now, let x denote an intermediate arithmetical quantity between A and C, and let y denote an intermediate arithmetical quantity between B and C. Then : If we can find values of x and y, so that $A : x :: y : B$, the mean proportional between x and y the *means*, will be equal to the mean proportional between A and B the *extremes*, and both equal to C, and the problem of finding two mean proportionals so that $A : x :: y : B$, will be solved.

Let A which denotes the area of the square G B H K circumscribed about the circle $X = 20$. Let C denote the area of the square A C E F: and, by hypothesis, let $\pi = \frac{25}{8} = 3{\cdot}125$.

Then:

$$\frac{A}{2} = \frac{20}{2} = 10 = C.$$

$$\therefore \frac{A}{\pi} = \frac{20}{3{\cdot}125} = 6{\cdot}4 = y.$$

$$\therefore \frac{y}{\frac{1}{4}A} = \frac{6{\cdot}4}{5} = 1{\cdot}28 : \text{ and, } 1{\cdot}28 : 1 :: 1 : \tfrac{1}{4}(\pi).$$

$$\therefore \frac{A}{1{\cdot}28} = \frac{20}{1{\cdot}28} = 15{\cdot}625 = x.$$

$$\therefore \frac{x}{\pi} = \frac{15{\cdot}625}{3{\cdot}125} = 5 = B.$$

$\therefore \{(C + \frac{1}{4} C) + \frac{1}{4}(C + \frac{1}{4} C)\} = \{(10 + 2{\cdot}5) + \frac{1}{4}(12{\cdot}5)\} = 12{\cdot}5 + 3{\cdot}125 = 15{\cdot}625 = x.$

And, $A : x :: y : B$. The product of x and y, the *means*, is equal to the product of A and B, the *extremes;* and it follows of necessity, that the mean proportional between x and y is equal to the mean proportional between A and B; and both = C; that is, $\sqrt{x \times y} = \sqrt{A \times B}$, or, $\sqrt{15{\cdot}625 \times 6{\cdot}4} = \sqrt{20 \times 5}$; and this equation or identity is equal to the area of an inscribed square to the circle X: and when $A = 20$, then, $C = 10$, therefore, $\frac{x}{y} = \left(\frac{x}{C}\right)^2$; and it follows, that $\frac{x}{y} = \left(\frac{\pi}{2}\right)^2$; therefore, $16\left(\frac{x}{y}\right) = (2\,\pi)^2$, and $A \times \frac{1}{4}(\pi) = 20 \times {\cdot}78125$; and this equation or identity $= 15{\cdot}625 = x$.*

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* A Mathematician of the De Morgan or Whitworth type, would catch at this and exclaim: *What! am I to argue with a man who would make the square* A C E F *and the circle* Y *equal in superficial area?* No doubt, it is self-evident, that the circle Y and the square A C E F cannot be equal in superficial area; and Mr. James Smith knows this as well as De Morgan or Whitworth, or any other "*recognised Mathematician.*" But, it is not self-evident that the circle Y and the square A B C D on the radius of the circle X cannot be equal in superficial area. We *know* that the radius of the circle Y is greater than the half of D C, a side of the square A B C D, and less than the half of A C a diagonal of that square, by construction.

It may be admitted, that this is a *particular case*, and quite unique, and consequently, that it is *true* only, when $A = 20$. But, by means of it, we discover the following formula for finding the area of a circle, from a given area of its inscribed square.

Let C denote the area of a square.

Then :

$(C + \frac{1}{4} C) + \frac{1}{4} (C + \frac{1}{4} C) =$ area of a circumscribing circle, and this holds good, whatever arithmetical value we may put

We also *know*, that if a circle equal in superficial area to the square A B C D exists, its radius must be greater than the half of D C and less than the half of A C. We also *know*, that if with D as centre and the half of A C as interval, we describe a circle, this circle will be an inscribed circle to the square A C E F.

It is self-evident, that the area of a square *may* or *may not* be represented by a square number. If represented by a square number, say 64, then, $\sqrt{64} = 8 =$ a side of the square, and is equal to the diameter of an inscribed circle: and the diameter of the circle is a finite arithmetical quantity. But, if the area of a square be represented by an arithmetical quantity that is not a square number, say 32, then, the diameter of an inscribed circle $= \sqrt{32}$. It may be admitted that $\sqrt{32}$ is a definite arithmetical expression, but, since we cannot extract the root without a remainder, it will surely not be argued by any "*recognised Mathematician*" or "*reasoning geometrical investigator*," that $\sqrt{32}$ represents a *finite* and *determinate* arithmetical quantity. Now, because the area of a circumscribing square to any circle is equal to twice the area of an inscribed square ; and because the area of an inscribed square to any circle is equal to twice the area of a square on the radius : it follows, that the mean proportional between the area of a square on the diameter of a circle and the area of a square on the semi-radius, is equal to the area of an inscribed square to the circle. This is axiomatic, if not self-evident, and the reason is obvious enough, viz. : The mean proportional between any given whole number and one-fourth part of it, is equal to half the given number; and it follows, that the double of any whole number, is equal to the half of four times that number.

Well, then, let the diameter of a circle $= 32$. Then : $\frac{32}{4} = 8 =$ semi-radius of the circle : and the mean proportional between the diameter and semi-radius $= \sqrt{32 \times 8} = \sqrt{256} = 16 =$ radius of the circle. But, when the area of a circumscribing square to a circle $= 32$, then, the diameter of the upon C; and it follows, that if C denote the area of a circle, then: $\{(C - \frac{1}{5} C) - \frac{1}{5} (C - \frac{1}{5} C)\}$ = area of an inscribed square to the circle; and again, the formula holds good, whatever arithmetical value we may put upon C.

Again:

If, in a right-angled triangle A B C, the sides A B and B C contain the right angle, and are in the ratio of 4 to 3, then, the hypothenuse A C will be 5, when the side A B = 4.

circle = $\sqrt{32}$: $\frac{1}{2}(\sqrt{32}) = \sqrt{8}$ = radius: and, $\frac{\text{radius}}{2} = \frac{\sqrt{8}}{2} = \sqrt{2}$ = semi-radius; therefore, the mean proportional between the diameter and semi-radius = $\sqrt{(\sqrt{32} \times \sqrt{2})} = \sqrt{(\sqrt{32 \times 2})} = \sqrt{(\sqrt{64})} = \sqrt{8}$: and is equal to radius; and $4\,(r^2) = 4(\sqrt{8})^2 = (4 \times 8) = 32$ = area of a circumscribing square to the circle. Let the diameter of the circle X = 1. Then: $\frac{1}{4} = \cdot 25$ = semi-radius of the circle X; and the mean proportional between the diameter and semi-radius $= \sqrt{1 \times \cdot 25} = \sqrt{\cdot 25} = \cdot 5$ = radius of the circle X. But, $16\,(sr^2) = 16\,(\cdot 25^2) = 16 \times \cdot 0625 = 1$ = area of the square G B H K, circumscribed about the circle X; and since the property of one circle is the property of all circles, it follows, that 16 times the area of a square on the semi-radius = area of a circumscribing square to every circle. Let the area of the square A B C D = 6·4. Then, $\sqrt{6.4}$ = D C the radius of the circle X. Now, $4\,(DC^2) = 4\,(6\cdot 4) = 25\cdot 6$ = area of a circumscribing square to the circle X, and it follows, that $\sqrt{25\cdot 6}$ = diameter of the circle X. But, the mean proportional between $\sqrt{6\cdot 4}$ and $\sqrt{25\cdot 6} = \sqrt{(\sqrt{6\cdot 4} \times \sqrt{25\cdot 6})} = \sqrt{(\sqrt{6\cdot 4 \times 25\cdot 6})} = \sqrt{(\sqrt{163\cdot 84})} = \sqrt{163\cdot 84} = 12\cdot 8$, and is not equal to the radius of the circle X, but equal to the area of the square A C E F inscribed in the circle X. Now, these are either facts or they are not facts. Facts they cannot be, if the value assigned to π by "*recognised Mathematicians*" is arithmetically correct. Well, then, let that *unscrupulous critic, and contemptible mathematical twaddler* De Morgan, go to work and prove that they are not facts, and consequently, that they are *mathematical* untruths. In this way he may demonstrate that *Mathematics as applied to Geometry by Mathematicians* is neither *a mockery, delusion,* nor *a snare,* and so establish his mathematical reputation, and put Mr. James Smith to silence. Let not that *vain man* imagine that he can convert truth into falsehood, by his *scurrilous abuse, contemptible ridicule, and ribald vulgarity !!!*

Hence:

$AC^2 : AB^2 :: \sqrt{20} : \sqrt{8{\cdot}192}$; that is, $25 : 16 :: \sqrt{20} : \sqrt{8{\cdot}192}$, and the product of the *means* is equal to the product of the *extremes;* that is, $16\,(\sqrt{20}) = 25\,(\sqrt{8{\cdot}192})$, and this equation or identity $= \sqrt{5120}$; but, since 5120 is not a square number, the mean proportional between the *means* and *extremes* is incommensurable. Now, $\frac{16}{25}(\sqrt{20}) = \sqrt{\frac{16^2}{25^2} \times 20} = \sqrt{\frac{256}{625} \times 20}$ $= \sqrt{{\cdot}4096 \times 20} = \sqrt{8{\cdot}192}$; $\frac{1}{2}(\sqrt{8{\cdot}192}) = \sqrt{2{\cdot}048}$; and, $\frac{1}{2}(\sqrt{2{\cdot}048}) = \sqrt{\frac{1^2}{2^2} \times 2{\cdot}048} = \sqrt{\frac{1}{4} \times 2{\cdot}048} = \sqrt{{\cdot}25 \times 2{\cdot}048} = \sqrt{{\cdot}512}$. But, $6{\cdot}25\,(\sqrt{2{\cdot}048}) = \sqrt{6{\cdot}25^2 \times 2{\cdot}048} = \sqrt{39{\cdot}0625 \times 2{\cdot}048}$ $= \sqrt{80}$; therefore, $\sqrt{(\sqrt{80}^{\,2} \times \sqrt{{\cdot}512}^{\,2})} = \sqrt{(80 \times {\cdot}512)} = \sqrt{40{\cdot}96}$ $= 6{\cdot}4$.

By hypothesis, let the circle Y and the square A B C D be equal in superficial area, and let the area of the circle $Y = 6{\cdot}4$

Then:

$2\,(6{\cdot}4) = 12{\cdot}8$ = area of the square A C E F inscribed in the circle X: $2\,(12{\cdot}8) = 25{\cdot}6$ = area of the square G B H K, circumscribed about the circle X: and $\frac{25{\cdot}6}{4} = 6{\cdot}4$ = area of the square A B C D, on the radius of the circle X.

Now:

$\sqrt{\frac{\text{Area of circle Y}}{\pi}} = \sqrt{\frac{6{\cdot}4}{3{\cdot}125}} = \sqrt{2{\cdot}048}$ = radius of the circle Y. $2\pi\,(\sqrt{2{\cdot}048}) = 6{\cdot}25\,(\sqrt{2{\cdot}048}) = \sqrt{6{\cdot}25^2 \times 2{\cdot}048}$ $= \sqrt{39{\cdot}0625 \times 2{\cdot}048} = \sqrt{80}$ = circumference of the circle Y; therefore, the mean proportional between the square of the circumference of the circle Y and the square of the radius $=$ $\sqrt{(\sqrt{80}^{\,2} \times \sqrt{2{\cdot}048}^{\,2})} = \sqrt{80 \times 2{\cdot}048} = \sqrt{163{\cdot}84} = 12{\cdot}8$; therefore, the mean proportional between the square of the circumference and the square of the radius of the circle Y is commensurable, and equal to the area of the square A C E F, and is therefore equal to twice the area of the square A B C D,

Again:

$\sqrt{\dfrac{\text{Area of circle Y}}{\pi}} = \sqrt{2{\cdot}048}$ = radius of the circle Y.

$2\pi\,(\sqrt{2{\cdot}048}) = \sqrt{80}$ = circumference of the circle Y; and it follows, that the mean proportional between the circumference and radius of the circle Y $= \sqrt{(\sqrt{80} \times \sqrt{2{\cdot}048})} = \sqrt{(\sqrt{80 \times 2{\cdot}048})} = \sqrt{(\sqrt{163{\cdot}84})} = \sqrt{12{\cdot}8}$; therefore, $\sqrt{12{\cdot}8}^{\,2} = 12{\cdot}8$ = area of the inscribed square A C E F, and is equal to twice the given area of the circle Y.

Again:

$\dfrac{\text{Circumference of circle Y}}{3{\cdot}125} = \dfrac{\sqrt{80}}{3{\cdot}125} = \sqrt{80} \div 3{\cdot}125^2 = \sqrt{\dfrac{80}{9{\cdot}765625}} = \sqrt{8{\cdot}192}$ = diameter of the circle Y; and it follows, that the mean proportional between the square of the circumference, and the square of the diameter of the circle Y $= \sqrt{(\sqrt{80}^{\,2} \times \sqrt{8{\cdot}192}^{\,2})} = \sqrt{(80 \times 8{\cdot}192)} = \sqrt{(655{\cdot}36)} = 25{\cdot}6$; therefore, the mean proportional between the square of the circumference, and the square of the diameter of the circle Y is commensurable, and is equal to four times the area of the square A B C D. But, $6{\cdot}25\,(\sqrt{2{\cdot}048}) = \sqrt{6{\cdot}25^2 \times 2{\cdot}048} = \sqrt{39{\cdot}0625 \times 2{\cdot}048} = \sqrt{80}$; therefore, $\sqrt{(\sqrt{80}^{\,2} \times \sqrt{{\cdot}512}^{\,2})} = \sqrt{(80 \times {\cdot}512)} = \sqrt{40{\cdot}96} = 6{\cdot}4$; and is equal to the area of the square A B C D on the radius of the circle X.

Again:

$\dfrac{\text{Circumference of circle Y}}{3{\cdot}125} = \sqrt{8{\cdot}192}$ = diameter, and it follows, that the mean proportional between the circumference and diameter of the circle Y $= \sqrt{(\sqrt{80} \times \sqrt{8{\cdot}192})} = \sqrt{(\sqrt{80 \times 8{\cdot}192})} = \sqrt{(\sqrt{655{\cdot}36})} = \sqrt{25{\cdot}6}$; therefore, $\sqrt{25{\cdot}6}^{\,2} = 25{\cdot}6$ = area of the square G B H K, circumscribed about the circle X, and is equal to four times the area of a square on the radius of the circle X.

Now, Mr. William Hepworth Dixon, where will you be if you attempt to work out similar results with the *mysterious* $\pi = 3{\cdot}14159265\ldots$? Who, my good Sir, is the "*impostor?*" That is the question! Is it J—— S——, the "*d——y;*" or is it the *mathematical* d——y of the *Athenæum?* Pray tell me?

I must bring this Letter to a close, which has run on so far towards "*fifteen quarto pages,*" as I dare say to make me, in the opinion of the WE of the *Athenæum*, *criminal* in the highest degree, if not guilty of a "*capital offence.*" Well, then, in conclusion, I may observe: I am not prepared to believe, that you, Mr. Wm. Hepworth Dixon, are either incompetent to "*read and understand,*" what I have written in this and my Letters of the 14th and 21st inst., or unable to comprehend the value and importance of the geometrical and mathematical truths I have brought under your notice; and I ask you, as a professed caterer of intellectual nourishment for the scientific public, will you not, if you decline to make these truths known to your numerous scientific readers, prove that the editorial WE of the *Athenæum* are Mathematical "*impostors?*"

I am, Sir,
Yours respectfully,
JAMES SMITH.

My early Letters to the Editor of the *Athenæum*, were intended to be suggestive rather than exhaustive. These Letters were obviously handed over to Professor de Morgan, and it would appear, that no argument would be suggestive of geometrical truth to that gentleman; and no reasoning, however cogent and logical, sufficient to convince him of the truth of the THEORY that 8 circumferences = 25 diameters in every circle. I believe, however, that Professor de Morgan knows the fact, that 8 circumferences are exactly equal to 25 diameters in every circle; but I do not believe that the learned Professor will ever have the candour to admit it. I have made some alterations and additions to this Letter:—If Professor de Morgan should happen to think I have done him an injustice, he can publish the original Letter and prove it. I have no doubt the following hint will enable him to lay his hands on the original Letter. "*His* (Mr. Smith's) *pamphlets and letters are all tied up together, and will form a curious lot when death or cessation of power to forage among bookshelves shall bring my* (Mr. de Morgan's) *little library to the hammer.*"

J. S.

In the Article, PSEUDOMATH, PHILOMATH, and GRAPHOMATH, Mr. A. de Morgan says:—"*There are some persons who feel inclined to think that Mr. Smith should be argued with: let those persons understand that he has been argued with, refuted, and has never attempted to stick a pen into the refutation.*" This was written on or about the 31st December, 1867. When or where has Mr. A. de Morgan ever "*refuted*," or attempted to refute, the geometrical truths brought under the notice of the Editor of the *Athenæum*, in my Letter of September 28th, 1867? Will he dare to *assert* that he did not receive a Letter from me, dated 30th June, 1864?* Will he dare to assert that the geometrical truths brought under his notice in that Letter, are not inseparably connected with those in my Letter of September 28th, 1867? When or where has he, or any other Mathematician, refuted these truths? Have I not refuted the Rev. Professor Whitwoth's false assertions, and fallacious arguments? Well, then, it appears to me almost inconceivable that so practiced a public writer as Mr. A. de Morgan should lay himself open to the charge of deliberate falsehood. Has he not done so?

BARKELEY HOUSE, SEAFORTH,
10th October, 1867.

TO THE EDITOR OF THE "ATHENÆUM."

SIR,

"*Here we are again, says the clown,*" but where will you be, Mr. Hepworth Dixon, if you "*read, mark, learn, and inwardly digest,*" this epistle? I may frankly confess that, in again addressing you, I hope to "*wheedle*" you into an examination of the properties of the remarkable geometrical figure represented by the enclosed diagram (*See Diagram XVI*), of which the following may be taken as the method of construction:—

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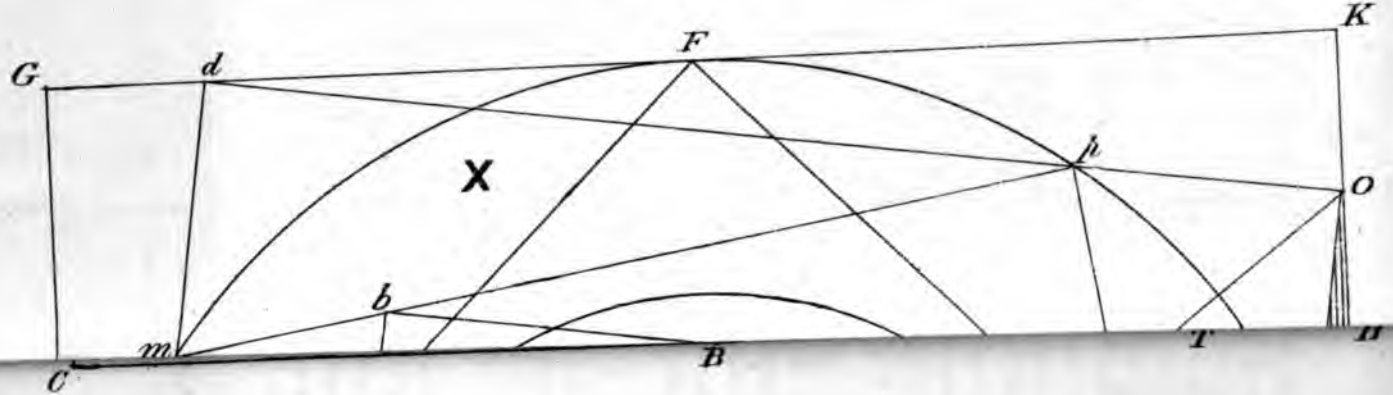
* See: *The British Association in Jeopardy, and Professor de Morgan in the Pillory, without hope of escape:* Page 78.

DIAGRAM XVI.



On the straight line A B describe the square A B C D, and draw the diagonal D B. On D B describe the square D B E F. With A as centre and A B as interval, describe the circle X, and about it circumscribe the square G C H K. From A B cut off a part A L equal to sixteen twenty-fifth parts of A B, and with A as centre and A L as interval, describe the circle Y. From the point L draw the tangent L M, making L M equal to three-fourth parts of A L, and join A M, producing the right-angled triangle A L M. From the point A, the centre of the circles, draw the straight line A N at right angles to A M, making A N equal to A M, and join N M. From the point N draw a straight line parallel to A B, to meet L M produced at the point P, producing the right-angled triangle N P M. On N M describe the square N M *a b*. Produce A N and A M, to meet the sides K H and H C of the circumscribing square G C H K, to the circle X at the points O and T, and join O T, producing the right-angled triangle O H T. On O T describe the square O T *c d*. The sides of the square O T *c d* cut the circumference of the circle X at the points *m*, *n*, *o*, and *p*. Join *m n*, *n o*, *o p*, and *p m*, producing the square *m n o p*, which is an inscribed square to the circle X, and therefore similar and equal to the square D B E F, on the diagonal of the square A B C D, the generating figure of the diagram. Join B M.

Now, B M is a straight line drawn from the right angle B in the triangle A B T, perpendicular to its opposite side AT, and connects the right angle in the triangle ABT with the obtuse angle M in the triangle N P M. The triangles A B T and A L M are similar right-angled triangles, and the sides containing the right angle are in the ratio of 3 to 4, by construction. The triangles N P M and O H T, are similar right-angled triangles, and in the triangle N P M the side N P = the sum of the two sides A L and L M which contain the right angle in the triangle A L M, and the side P M = the difference of A L and L M. In the triangle O H T the side O H = the sum of the two sides A B and B T, which contain the right angle in the triangle A B T, and the side H T = the difference of A B and B T. Hence: The sides which contain the right angle in the similar triangles, N P M and O H T, are in the ratio of 7 to 1.

Hence:

(A L + L M) = N P, and, (A L — L M) = P M ; therefore, (A L^2

$+ LM^2 + AM^2) = (NP^2 + PM^2) = NM^2 = 3\frac{1}{8}(AL^2)$ = area of the square N M *a b.*

$(AB + BT) = OH$, and, $(AB - BT) = HT$; therefore, $(AB^2 + BT^2 + AT^2) = (OH^2 + HT^2) = OT^2 = 3\frac{1}{8}(AB^2)$ = area of the square O T *c d.*

$(NP + PM)^2$ = area of a circumscribing square to the circle Y, and, $(NP^2 - PM^2)$ = area of a regular inscribed dodecagon to the circle Y.

$(OH + HT)^2$ = area of the square G C H K circumscribed about the circle X, and, $(OH^2 - HT^2)$ = area of a regular inscribed dodecagon to the circle X.

In a right-angled triangle, if a perpendicular be drawn from the right angle to the opposite side, the triangles on each side of it, are similar to the whole triangle and to each other. (Euclid: Prop. 8, Book 6).

Now, in the triangle A B T, B M is a straight line drawn from the right-angle B, perpendicular to its opposite side A T; therefore, A M B and B M T, the triangles on each side of B M, are similar right-angled triangles and similar to the whole triangle A B T. But, in the right-angled triangle A M B, M L is a perpendicular drawn from the right angle M to the opposite side A B; therefore, A L M and M L B, the triangles on each side of M L, are similar right-angled triangles, and similar to the whole triangle A M B. But, the triangles A M B and B M T are similar triangles; therefore, the triangles M L B and B M T are similar triangles; therefore, the five triangles A B T, A L M, A M B, B M T, and M L B are similar right-angled triangles, and the sides containing the right angle are in the ratio of 3 to 4. But, B M is common to the two right-angled triangles M L B and B M T, and is the hypothenuse of the former and the perpendicular of the latter; therefore, $(ML^2 + LB^2) = (BT^2 - MT^2)$.

Now, π denotes the number of times the diameter of a circle is contained in the circumference: or in other words, π denotes the circumference of a circle, when the diameter is taken as unity, and the perimeter of a circumscribing square to a circle of diameter unity = 4; therefore, $\frac{\pi}{4}$ expresses the ratio between the circumference of a circle and the perimeter of its circumscribing square. Now, $4(8) = 10(3{\cdot}2) = 32$ = the perimeter of a circumscribing square to a circle of diameter 8; therefore, $8\pi =$ the circumference of a circle, when the perimeter of a circumscribing square $= 32$, whatever be the value of π. For example: by hypothesis, let $\pi = 3{\cdot}1416$, and let the diameter of a circle $= 8$. Then: $8\pi = 8 \times 3{\cdot}1416 = 25{\cdot}1328 =$ circumference of the circle; and $\pi : 4 :: 25{\cdot}1328 : 32$, the perimeter of a circumscribing square; and it is obvious that any other finite hypothetical value of π, intermediate between 3 and 3·2, would produce a similar result.

By means of the following algebraical formula, we obtain some remarkable results:—

$$\frac{A}{\pi} = y; \ \left(A \times \frac{\pi}{4}\right) = x; \text{ and, } \frac{x}{\pi} = B.$$

Now, let A denote the area of a square circumscribed about a circle, and let B denote the area of a square on the radius of the circle. Let x and y denote the areas of circles, so that x is less than A, and y greater than B, and let A be given to find two mean proportionals, so that $A : x :: y : B$.

Let $A = 32$, and, by hypothesis, let $\pi = 3{\cdot}2$.

Then:

$$\frac{A}{\pi} = \frac{32}{3{\cdot}2} = 10 = y : \left(A \times \frac{\pi}{4}\right) = 32 \times {\cdot}8 = 25{\cdot}6 = x :$$

and, $\frac{x}{\pi} = \frac{25{\cdot}6}{3{\cdot}2} = 8 = B$; therefore, $A : x :: y : B$; that is, $32 : 25{\cdot}6 :: 10 : 8$; and the product of the *extremes* is equal to the product of the *means;* therefore, the mean proportional between A and B, is equal to the mean proportional between x and y; that is, $\sqrt{32 \times 8} = \sqrt{25{\cdot}6 \times 10} = \sqrt{256} = 16 = \frac{A}{2}$, or, (2 B.)

Again: Let $A = 32$, and by hypothesis, let $\pi = 3{\cdot}125$. Then: $\frac{A}{\pi} = \frac{32}{3{\cdot}125} = 10{\cdot}24 = y : \left(A \times \frac{\pi}{4}\right) = 32 \times {\cdot}78125 = 25 = x$: and, $\frac{x}{\pi} = \frac{25}{3{\cdot}125} = 8 = B$; therefore, $A : x :: y : B$; that is, $32 : 25 :: 10{\cdot}24 : 8$; and the product of the *extremes* is equal to the product of the *means;* therefore, the mean proportional between A and B, is equal to the mean proportional between x and y; that is, $\sqrt{32 \times 8} = \sqrt{25 \times 10{\cdot}24} = \sqrt{256} = 16 = \frac{A}{2}$, or, (2 B.)

In both examples the mean proportional between A and B, and x and y, $=$ area of an inscribed square to a circle of which A denotes the area of a circumscribing square: but neither affords any proof of the true value of π. Either hypothetical value of π on which the foregoing calculations are worked out, may be right, or both may be wrong, but both cannot be right. Hence: we must resort to some other means to prove whether either or neither is the true arithmetical value of π.

Let $A = 32$, and, by hypothesis let $\pi = 3$.

Then.

$\frac{A}{\pi} = \frac{32}{3} = 10\frac{2}{3} = y : (A \times \frac{\pi}{4}) = 32 \times {}^{\cdot}75 = 24 = x$: and, $\frac{x}{\pi} = \frac{24}{3} = 8 = B$; therefore, $A : x :: y : B$; that is, $32 : 24 :: 10\frac{2}{3} : 8$; and the product of the *extremes* is equal to the product of the *means*; therefore, the mean proportional between A and B, is equal to the mean proportional between x and y; that is, $\sqrt{32 \times 8} = \sqrt{24 \times 10\frac{2}{3}}$; and this equation or identity $= \sqrt{256} = 16 = \frac{A}{2}$ or, 2 (B). Hence: $4\left(\frac{\pi}{2}\right)^2 = \pi^2$, whatever be the value of π. For example: $4\,(1{\cdot}5^2) = 4 \times 2{\cdot}25 = 9 = 3^2$, when we assume $\pi = 3$. How, then, can the arithmetical value of π be an indeterminate arithmetical quantity? Surely these facts ought to suggest to a "*recognised Mathematician*" like Professor de Morgan, that there is something more in Geometry, *than was ever dreamed of in his mathematical philosophy.*

Well then, referring to the diagram (See Diagram XVI.), let A denote the area of the square A B C D on the radius of the circle X. Let B denote the area of the square N M *a b* on the hypothenuse of the right-angled triangle N P M. Let C denote the area of the square *mnop*, or, D B E F, either of which is an inscribed square to the circle X. Let D denote the area of the square O T*cd* on the hypothenuse of the right-angled triangle O H T. And let E denote the area of the square G C H K circumscribed about the circle X.

Then: A : B :: D : E; and the product of the *extremes* is equal to the product of the *means*; that is, $A \times E = B \times D$; therefore, the mean proportional between A and E = the mean proportional between B and D, and both = C, which denotes the area of the inscribed squares to the circle X.

Hence:

$\frac{E}{4} = A : (AL^2 + LM^2 + AM^2) = 3\frac{1}{8}\,(AL^2) = B : \frac{E}{2} = 2A$

$= C : (AB^2 + BT^2 + AT^2) = 3\frac{1}{8}(AB^2) = D : (2\,AB)^2 = E$: and $\frac{7}{8}(A)$ = the difference between D and E : and it follows that :—If $A = 16$: then, $B = 20{\cdot}48$: $D = 50$: and $E = 64$; therefore, $A : B :: D : E$; that is, $16 : 20{\cdot}48 :: 50 : 64$. The product of the means = the product of the extremes; and it follows, that the mean proportional between the means = the mean proportional between the extremes; that is, $\sqrt{20{\cdot}48 \times 50} = \sqrt{16 \times 64} = \sqrt{1024} = 32 = C$.

Now, let *m* and *n* denote the areas of the circles X and Y, and let the arithmetical value of *n* be any finite quantity, say 32, and be given to find B, that is, to find the arithmetical value of the area of the square N M *a b*. Then: $\sqrt{\frac{32}{3{\cdot}125}} = \sqrt{10{\cdot}24} = AL$ the radius of the circle Y, which is also the perpendicular of the right-angled triangle A L M; and $AL : LM :: 4 : 3$, by construction; therefore, $\frac{3}{4}(AL) = \frac{3}{4}(\sqrt{10{\cdot}24}) = \sqrt{\left(\frac{3^2}{4^2} \times 10{\cdot}24\right)} = \sqrt{\left(\frac{9}{16} \times 10{\cdot}24\right)} = \sqrt{({\cdot}5625 \times 10{\cdot}24)} = \sqrt{5{\cdot}76} = LM$ the base of the right-angled triangle A L M, and A L and L M contain the right angle; therefore, $(AL^2 + LM^2) = (10{\cdot}24 + 5{\cdot}76) = 16 = AM^2$; therefore, $(AL^2 + LM^2 + AM^2) = 3\frac{1}{8}(AL^2) = 32 = B$; that is, = area of the square N M *a b*. For, $AN = AM$, by construction, and A N and A M contain a right angle, A; therefore, $(AN^2 + AM^2) = (16 + 16) = 32 = NM^2 = B$; that is, = area of the square N M *a b*, and demonstrates that *n* and B are exactly equal; that is, that the circle Y and the square N M *a b* are exactly equal in superficial area. And similarly, from a given value of *m*, we may find the arithmetical value of D, that is, the arithmetical value of the area of the square O T *c d*, and so prove that *m* and D are exactly equal in superficial area.

Now, Sir, to prove that *n* is not equal to B, and *m* not equal to D, you must demonstrate the following things to be geometrical absurdities :—

First: $\frac{7}{8}(A)$, that is, seven-eighths of the area of the square A B C D, the generating figure of the diagram, = the difference between D and E.

Second: The sum of the areas of the four triangles *b p* N, N *o* M, M *n a*, and *a m b*, about the square N M *a b*, = the difference between B and C.

Third: The sum of the areas of the four triangles *m c n*, *n* T *o*, *o* O *p*, and *p d m*, about the square *m n o p*, = the difference between C and D.

Fourth: The parts of the circle X, contained by the sides of the square *m n o p* and their subtending arcs, are equal to the parts of the square O T *c d*, about the square *m n o p*, each to each.

Fifth: The sides that contain the right angle, in the triangles N P M and O H T, are in the ratio of 7 to 1, by construction.

Hence:

The area of every circle is equal to the area of a square on the hypothenuse of a right-angled triangle, of which the sides that contain the right angle are in the ratio of 7 to 1, and their sum equal to the diameter of the circle. Corollary: The area of every circle is equal to the sum of the areas of squares on the sides of a right-angled triangle, of which the sides that contain the right angle are in the ratio of 3 to 4, and the longer of these sides the radius of the circle.

These geometrical truths are very far from exhausting the properties of the remarkable figure represented by the diagram, but I am warned that it is time to bring this Letter to a close, as it is fast running on to "*fifteen quarto pages.*" Well, then, in conclusion, if the WE of the *Athenæum* find it impossible to controvert the geometrical and mathematical truths I have now brought under their notice, and yet decline to give currency to them through the columns of the leading Scientific Journal, I leave it to their own consciences to answer the following question:—What will they be?

I am, Sir,

Yours respectively,

JAMES SMITH.

BARKELEY HOUSE, SEAFORTH,
12th October, 1867.

TO THE EDITOR OF THE "ATHENÆUM."

SIR,

I posted a Letter to your address yesterday, and omitted to enclose the diagram, which I send you herewith.

In my Letter I stated that the truths to which I addressed your attention were far from exhausting the properties of this remarkable geometrical figure, and I may take this opportunity of calling your attention to the following facts, of which, as a Geometer and Mathematician, you may readily convince yourself.

The sides of the squares D B E F on the diagonal of the square A B C D, intersect and bisect the sides of the square O T *c d* on the hypothenuse of the right-angled triangle O H T.

The side *a* M of the square N M *a b*, intersects and bisects D B the diagonal of the square A B C D, at the point *e*, therefore, the point *e* is the centre of the square A B C D, the generating figure of the diagram. With A as centre and A *e* as interval, describe a circle. This will be an inscribed circle to the squares D B E F and *m n o p*.

The sides of the square N M *a b*, on the hypothenuse of the right-angled triangle N P M, cut the circumference of the circle Y at eight points, by means of which we may produce two inscribed squares to the circle Y.

How could these things be, my good Sir, if there were no definite relations existing between the circles, squares, and triangles, of which the diagram is composed?

I am, Sir,
Yours respectfully,
JAMES SMITH.

At this time, and in consequence of the *scurrilous* attack on my Letter to His Grace the Duke of Buccleuch, I decided upon writing a series of Letters to the Editor of the *Athenæum*, with a view to publication, and in this way bring under the notice of the Mathematical world the Geometry of the Circle; but I was diverted from this intention by the correspondence referred to in the early part of my pamphlet "*Euclid at Fault*," which commenced on the 17th December, 1867, and extended over a period of nearly six months. In the article: PSEUDOMATH, PHILOMATH, AND GRAPHOMATH, Mr. A. De Morgan admits the receipt of, and thanks the Editor of the *Athenæum* for, nine of these Letters. But not a word about Mr. Smith's Letters of November 7th, 20th, 23rd, 27th; December 2nd, 5th, 7th, 10th, and 16th. (One of these Letters (November 20th) concludes as follows:—"*Now, Mr. Editor, whether you do, or do not, think this communication worth your attention, pray let the learned Professor De Morgan have the opportunity of perusing it.*") Does Mr. A. De Morgan mean to say that these Letters never came into his hands? Mr. Smith is of opinion that all of them came into the possession of that gentleman, and if Mr. Smith is wrong, let Mr. A. De Morgan speak plainly, say so, and prove it! That of December 10th, appears in my correspondence with the Rev. Professor Whitworth. When Mr. A. De Morgan has his next "*bit of sport*" with the "*great pseudomath of his time*," let him prove that he saw for the first time the pseudomath's effort of December 10th, in that correspondence. Well then, it is quite impossible that I can introduce into this work, all my Letters to the Editor of the *Athenæum*, which would make a volume of themselves, and I shall content myself with giving three of them:—

JAMES SMITH *to* WM. HEPWORTH DIXON, ESQ., EDITOR OF THE "ATHENÆUM."

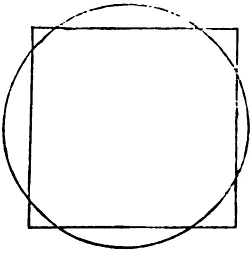
BARKELEY HOUSE, SEAFORTH,
7th November, 1867.

SIR

The supplement No. 10, to the "*Budget of Paradoxes*," by Professor De Morgan, which you will find in the *Athenæum* of December 1, 1866, commences thus:—"*And now for an episode on* *the things of the day. My account of Mr. Thom and* 666, *appeared on October* 27 *: on the* 29*th, I received from the Editor a copy of Mr. Thom's Sermons, published in* 1863 *(he died, February* 27*th,* 1862*), with best wishes for my health and happiness. The Editor does not name himself in the book; but he signed his name in my copy : and may my circumference never be more than* 3⅛ *of my diameter, if the signature, name, and writing both, were not that of my* ⊙ □*ing friend, Mr. James Smith.*" The Budget ends with the following words: "*His (Mr. Thom's) fame must rest on his senary tripod.*"

Well, Mr. Editor, it is now my turn for "*an episode on the things of the day.*" You know, Sir, that I am a Member of the Mersey Docks and Harbour Board, and Thursday is our Board day. The Board is a public one, and short-hand writers attend regularly to report the proceedings. One of these gentlemen, on entering the Board-room to-day, handed me a printed paper. I wish you could have seen the air with which he did it; I can assure you that Professor De Morgan himself could not have made me such a presentation with a greater air of triumph. The following is a copy of the paper:—

"THE CIRCLE SQUARED.



Square Root .... ·88622692545527.
Area .... ·78539816339734548309993729
Circumference of circle.........3.14159265358938193239974916.
Ratio ....3·92699081698672741549968645.

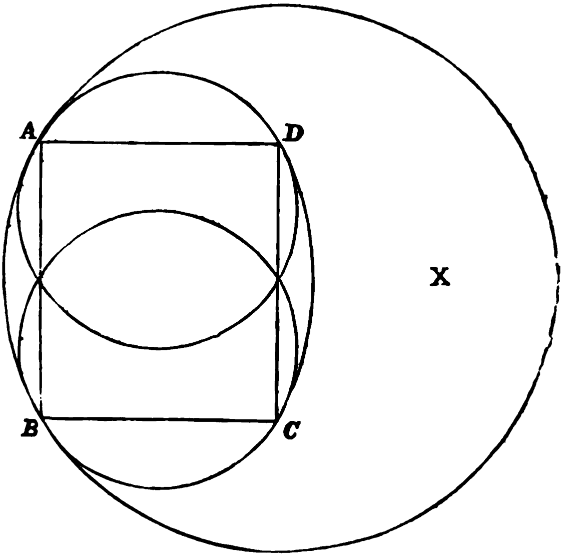
1·25

(Signed) Yours truly,

H. H——."

Within five minutes, I threw off, and presented Mr. H. with a paper, of which the following is a copy:—

DIAGRAM (Fig. 1.)



Thank you, Mr. H——. Since you are competent to calculate the ratio of diameter to circumference in a circle to 26 places of decimals, you will, of course, be competent to solve a much simpler problem. Well then, in the above geometrical figure, you have a perfect ellipse about the rectangle A B C D. Find the ratio between the periphery of the ellipse, and the circumference of the circle X.*

On the Reporter's leaving, I handed Mr. H—— another paper, and I afterwards saw him outside the Board-room and had a few minutes' conversation with him. The following is a copy of the second paper:—

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* Mr. H. H——, the circle squarer, is the father of the Mr. H—— referred to in this paragraph; but at the time of penning it I was not aware of this fact.

In this geometrical figure, let A B C be a right-angled triangle, and A B the radius of the circle, and let the sides A B and B C,

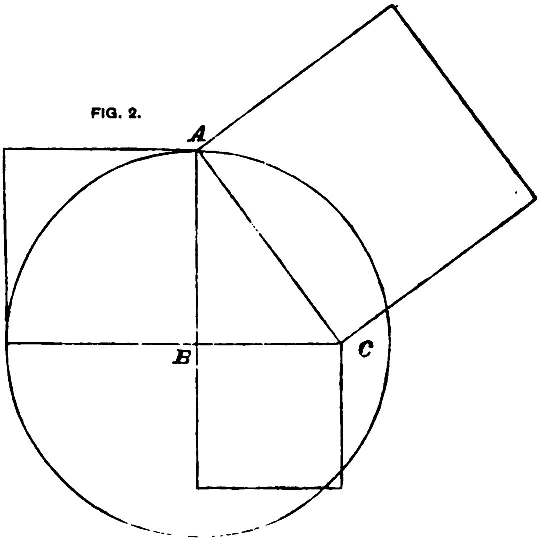

FIG. 2.

which contain the right angle in the triangle A B C, be in the ratio of 4 to 3, by construction.

Then: If A B = 1, $\frac{3}{4}$(A B) = ·75 = B C, and A B^2 + B C^2 = 1^2 + ·75^2 = 1 + ·5625 = 1·5625 = A C^2; therefore, $\sqrt{1{\cdot}5625}$ = 1·25 = A C; and the value of A C is one of the terms of your ratio, and 5 times the area of a circle of diameter unity, is the other term. But, (A B^2 + B C^2 + A C^2) = (1 + ·5625 × 1·5625) = 3·125, is the sum of the areas of the squares on the sides of the triangle A B C; and on the THEORY that 8 circumferences of a circle are exactly equal to 25 diameters, is equal to the area of the circle. Now, the area of a circle of radius 1. is represented by the same arithmetical symbols as π, whatever be the value of π.

Again: If A B = $\frac{1}{2}$ = ·5, then, $\frac{3}{4}$ (A B) = ·375 = B C; and A B^2 + B C^2 = ·5^2 + ·375^2 = ·25 + ·140625 = ·390625 = A C^2; therefore, $\sqrt{{\cdot}390625}$ = ·625 = A C, and 5 times A C = 3·125. But, (A B^2 + B C^2 + A C^2) = ·25 + ·140625 + ·390625) = ·78125, is the sum of the areas of the squares on the sides of the triangle A B C, and is equal to $\frac{3 \cdot 125}{4}$; that is, = to one fourth part of the area of the circle when A B = 1; and on the THEORY that 8 circumferences = 25 diameters in every circle = area of a circle of diameter unity.

Now, the square root of ·78125 is ·8838..., and in your paper you give the square root of the area of a circle of diameter unity = ·88622692545 27 . . . But, pray, my good Sir, what on earth has either the one or the other to do with the ratio of diameter to circumference in a circle? Are you aware that by your paper you make one of the terms of your ratio equal to 5 times the area of a circle of diameter unity, and the other equal to 5 times the semi-radius of a circle of diameter unity? Are you aware that these would have been the terms of your ratio had you given me 260 decimals instead of 26?

Now, Sir, when the diameter of a circle is represented by unity, then, on the THEORY that 8 circumferences of a circle are exactly equal to 25 diameters:—

Area of the circle = ·78125.
Circumference of the circle = 4 (·78125) = 3·125.
5 times area = circumference + area = 3·125 + ·78125 = 3·90625.
5 times semi-radius = 5 × ·25 = 1·25.

Therefore, $\frac{3 \cdot 90625}{1 \cdot 25}$ expresses the ratio between the circumference and diameter of the circle.

But, the two terms of a ratio may be divided by the same arithmetical quantity without altering the ratio itself. Hence: If we divide the two terms of the ratio $\frac{3 \cdot 90625}{1 \cdot 25}$ by 1·25, we obtain the equivalent ratio $\frac{3 \cdot 125}{1}$, which is the true arithmetical expression of the ratio between the circumference and diameter in every circle.

Hence: When A B the radius of the circle = 1, then 5 (A C) = 5 × 1·25 = 6·25 = circumference of the circle; and 12·5 $\left(\frac{AB}{2}\right)^2$ = 12·5 (·5²) = 12·5 × ·25 = 3·125 = area of the circle, and since the property of one circle is the property of all circles, it follows of necessity, that 12½ times the area of a square on the semi-radius = area in every circle.

When I spoke to Mr. H—— outside the board-room, I soon found that the elliptical figure was a perfect puzzle to him. I then said:—I may tell you, Mr. H——, that before you can controvert these facts, you must establish some relation between your ratio, and a circle standing in connection with some other geometrical figure or figures. Pray make the attempt, my good Sir, and where will you be? I may tell you that when you are prepared to teach me anything in Geometry and Mathematics, you will not find me taking my stand in the ranks of that numerous class "*who despise wisdom and instruction.*"

Now, Mr. Editor, we are not without a recognised Mathematical authority in Liverpool, and it so happened, that (after my little episode with Mr. H——), I had occasion to see this gentleman on a matter of business, and referred to it. I drew the elliptical figure (See Fig. 1.) which he admitted was new to him. I then asked him to find the ratio between the periphery of the ellipse and circumference of the circle X. He saw in a moment that AB is a chord of an arc of 60°, and very soon found the ratio between the periphery of the ellipse and circumference of the circle to be as 240 to 360, or as 2 to 3. True, said I, and it follows that the periphery of the ellipse is exactly equal to four third parts of the circumference of one of the small circles, and its longer diameter or major axis equal to 3 times the radius of the small circles. This he admitted to be true, and I put the question:—How could this be possible if there were no definite relation between the diameter and circumference of a circle? And I eventually drew from him the admission, that it would be impossible. What, Mr. Editor, will you and Professor de Morgan say to this?

Now, Sir, suppose me to construct a diagram made up of Fig. 1 and Fig. 2 in combination, would it not be a "*senary tripod?*" It would be a queer three-legged stool, no doubt, but employing language somewhat after the fashion of your funny friend De Morgan, it would be a "*senary tripod,*" and a tripod that renders its own oracle. If this be nonsense, let you and Professor de Morgan prove it.

I have no doubt that you, Mr. Wm. Hepworth Dixon, was one of that singularly distinguished assembly of literary and scientific men, who, on Saturday last, were drawn together by the farewell banquet, given to Mr. Charles Dickens; and so heard the brilliant speech of the noble Chairman, Lord Lytton, who in proposing the toast of the evening, is reported to have said:—"*You are now invited to do honour to a kind of royalty, which is seldom very peacefully acknowledged until he who wins and adorns it has ceased to exist in the body, and is unconscious of the empire which his thoughts have bequeathed to his name. Happy is the man who makes clear his title deeds to the royalty of genius, while he yet lives to enjoy the gratitude and the reverence of those he has subjected to his sway. Though it is by conquest that he achieves his throne, he, at least, is a conqueror whom the conquered bless, and the more despotically he enthrals, the dearer he becomes to the hearts of men. Rarely, I say, is that kind of royalty quietly conceded to any man of genius, till his tomb becomes his throne. Yet none of us think it strange that it is granted without a murmur to the guest we receive to-night.*" Happy, indeed, in his literary career, should that man be, of whom Lord Lytton, himself a poet, novelist, dramatist, critic, orator, statesman, and philosopher, could so speak. Mr. Dickens is said to have replied with much emotion, and in the course of his address observed:—"*I have always tried to be true to my calling—never unduly to assert it, on the one hand, and never on any pretence or consideration to permit it to be patronised in my person, on the other—this has been the steady endeavour of my life; and I have occasionally been vain enough to hope, that I may leave its social position better than I found it.*" He subsequently said:—"*And here, in reference to the inner circle of writers, and the outer circle of the public, I feel it a duty to-night to offer two remarks. I have in my day, at odd times, heard a great deal about literary sets, and cliques, and coteries, and barriers; about keeping this man up, and that man down; and about sworn disciples, and sworn unbelievers, about mutual admiration societies, and I know not what other dragons in the upward path. I began to tread it when I was very young, without influence, without money, without companion, introducer, or adviser, and I am bound to put in evidence in this place, that I never lighted on these dragons yet. So I have heard in my day, at other odd times, much generally to the effect, that the English people have little or no love of art for its own sake, and that they don't greatly care to acknowledge or do honour to the artist. My own experience has* *uniformly been exactly the reverse. I can say that of my countrymen, if I cannot say that of my country.*"

Happy, aye, proud, and justly proud, may that man be, who in his literary career has been so fortunate as to be able so to speak; and yet, judging from the last words of my quotation from Mr. Dickens's speech, one thing is still wanting to complete his happiness. Mr. Dickens can say more than I can. In my literary career, those who have had a word of commendation for me are few indeed. My *upward path* has been beset by "*dragons*" on every side. I have "*lighted*" on them among "*literary sets*," and the vials of their abuse and ridicule have been poured forth through the columns of our leading literary and scientific journals. I have "*lighted*" on them among the "*cliques*" who form our Literary and Philosophical Societies! I have "*lighted*" on them in that *mutual admiration Society*, "*The British Association for the Advancement of Science.*" The "*Dragons*" of that body have the power, and for years have made a point of exercising it, to prevent geometrical truth from being brought to light through that channel. I have come into conflict with these "*dragons*," from time to time, individually and personally! They may have occasionally wounded me, but they have not yet succeeded in inflicting such a wound, that I should say with Mercutio:—"*'Tis not so deep as a well, nor so wide as a church door, but 'tis enough, 'twill serve.*"

November 9th.

One of these dragons has given me a name, and another has made a nondescript of me, and fixed me a location somewhere between earth and heaven. You, Mr. Editor, will be proud to hear what the latter dragon says of you to-day. Pray read the following:—

"MR. HEPWORTH DIXON."

"Our vein, as our readers well know, is rarely anticipatory. If the Liverpool Institute had inveigled even royalty within its walls, we should have been silent till after the event. It is not the Prince of Wales who is to present the Institute prizes next Thursday, but the King of English critics, the Editor of the *Athenæum*. We, of Liverpool, are rarely visited by great literary men, and there is little fear that we should be drawn from the gilded gods of our own cult to worship unduly the less gorgeous divinities of the book world. *Porcupine* will not be suspected, therefore, of snobbery or adulation, when he calls upon his townsmen to do honour to one of the most eminent and most deserving of living English writers. Mr. Hepworth Dixon has the perceptive imagination and interpretative genius of a true critic, and he unites to these qualities those more brilliant and vivacious gifts, without which, critics, however great in the judgment-seat, can hardly be distinguished as literary creators. Let us do him honour, therefore, and prove that even at Liverpool, the guinea gold of intellect may pass current, without the stamp of rank."

What must I do next Thursday? You know, Mr. Editor, that it was my intention to be present at the distribution of prizes of the Liverpool Institute. How can I shew my face there now? In the opinion of *Porcupine*, and Professor De Morgan, I should be but as one of a swarm of flies about an elephant. I think, however, I shall make an attempt to be present on that interesting occasion. Of one thing I am certain. You, Mr. Hepworth Dixon, will acquit me of being guilty either of "*snobbery or adulation*," should I be found at the Liverpool Institute on the evening of Thursday next.

Well! Well! "*All's well, that ends well.*" It may be that my *tomb* may become my *throne*, or, it may be, that before I "*cease to exist in the body*," I may be privileged to overcome and rid my path of the "*dragons*" that have hitherto beset it; but whether this should or should not be my good fortune, I shall pursue the even tenor of my way, without any anxiety as to the result. What is truth? On more subjects than one, truth is made a mere matter of opinion. But, Geometrical truth can never be made a matter of opinion; and with regard to it I ask no man's opinion. In your opinion, Mr. Editor, "*an expert arithmetician such as is Mr. J. Smith, may fancy that calculation, merely as such, is Mathematics.*" * In your opinion, Mr. J. Smith has shewn himself "*utterly destitute of all that distinguishes the reasoning geometrical investigator from the calculator.*"* In Mr. J. Smith's opinion, you have failed to establish the *truth* of either of your opinions. In your opinion Mr. J. Smith is a d——y. Mr. J. Smith is of a different opinion. But what matters it what our opinions are on these points?

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* See *Athenæum*, May 11, 1861. "Review: The Quadrature of the Circle: Correspondence between an Eminent Mathematician and James Smith, Esq.

The question still remains, what is geometrical truth? The answer to this question is not a matter of opinion. A geometrical theorem is based on *indisputable* data, and consequently, can be logically reasoned out to an *indisputable* conclusion.

Well, then, from time to time, I shall bring to light through the press, the geometrical truths it has been my privilege to discover, as opportunity offers, and as it suits my convenience, (and you know, Mr. Editor, it is with this object in view that I am making use of you just now,) confident that neither you, nor your friend Professor De Morgan, nor even the combination of all the talents represented by the British Association, can for any lengthened period succeed in convincing the world, that truth should be called by another name; and, that Geometry is not an exact science, but a mere "*mockery, delusion, and a snare.*"

I am, Sir,
Yours respectfully,
JAMES SMITH.

TO W. HEPWORTH DIXON, ESQ., EDITOR OF THE "ATHENÆUM."

BARKELEY HOUSE, SEAFORTH,
15*th November*, 1867.

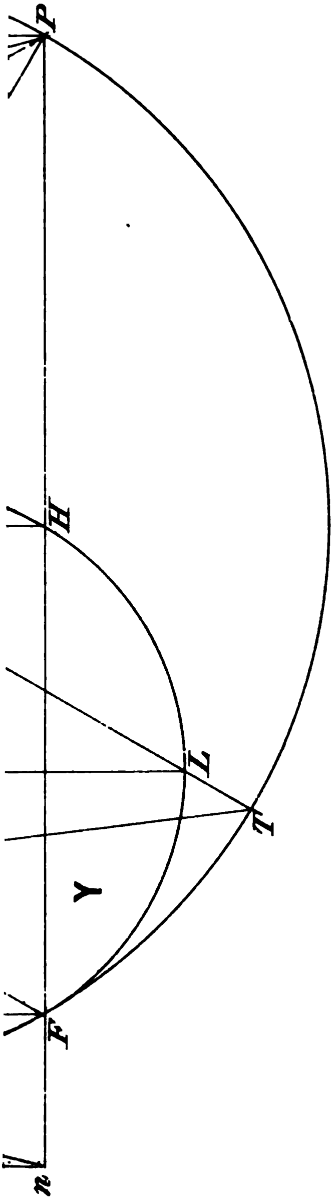
SIR,

Although very unwell, I found my way to the Liverpool Institute last evening, and heard your brilliant and elegant address; and I can honestly assure you that, you had not a more attentive or delighted listener. It was to me an evening that I shall never forget.

Now, my dear Sir, after hearing the noble sentiments to which you gave utterance last night, I cannot imagine it possible that you can intentionally lend yourself, and the influence of the *Athenæum*, to *write* down geometrical truth. Now, there appeared in the *Athenæum*, of September 14th, a short notice of my Letter to His Grace the Duke of Buccleuch, on the Quadrature and Rectification of the Circle, obviously from the pen of Professor de Morgan, which drew from me two Letters addressed to the Editor. These Letters were evidently handed over to that gentleman, and drew from him the scurrilous and untruthful trash, which appeared in "*Our Weekly Gossip*" of September 28th; and I cannot help thinking that my Letters must have been handed to Professor de Morgan by one of your Sub-Editors without your knowledge, and that the article referred to got into the *Athenæum* without your consent. I cannot imagine that he, whom I heard, and whom I looked upon, last evening, would ever have intentionally permitted such trash to disgrace the columns of the leading Scientific Journal. If I am right in these impressions, all I can say is, that I shall only be too happy to retract anything I may have said that reflects upon yourself personally.

Now, Sir, in my Letter to you of the 7th inst., I have shewn by means of a very simple geometrical figure, how a perfect ellipse or oval may be constructed, of which the periphery is exactly equal to two-third parts of the circumference of one circle, and exactly equal to four-third parts of two other circles, and the longer diameter or major axis of the ellipse or oval exactly equal to three times the radius of the small circles; and it follows of necessity, that the longer diameter of the ellipse is exactly equal to three-fourth parts of the diameter of the large circle. These facts are now admitted by a gentleman, as competent to deal with the subject as any man in England—not excepting Professor De Morgan—and who, for many years, was one of my most resolute opponents. He now says, and *admits*, that these facts could not be possible, if there were no definite relation between the diameter and circumference of a circle.

The following may be taken as the method of construction of the enclosed diagram. *(See Diagram XVII.)* Let A B be a straight line. Then: with A as centre and A B as interval, describe the circle X, and with B as centre and B A as interval, describe the circle Y. The circumferences of these circles cut each other at the points C and D. Join C D. Draw E F, and G H, perpendicular to C D, and therefore parallel to A B, and join E G and F H, producing the rectangle E F H G. With D as centre, and D E as interval, describe the circle Z, and with C as centre, and C G as interval, describe the arc G N H, producing the ellipse or oval about the rectangle E F H G. Produce A B and C D,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both ways, to meet the periphery of the oval or ellipse, at the points K, L, M, and N, and join K C and K D, producing the equilateral triangle K C D to the circle X. From the angle E in the rectangle E F H G draw E P, a diameter of the circle Z, and at right angles to E P draw the diameter S T. From the angle F in the rectangle E F H G draw a straight line through the point A, the centre of the circle X, to meet and terminate in the circumference of the circle Z, at the point V. Produce F H to meet and terminate in the circumference of the circle Z, at the point P, and join V P, producing the equilateral triangle V F P to the circle Z. The lines K C and E A intersect and bisect each other at the point R. Join T R, S R, and S P. The side V F, in the equilateral triangle V F P, cuts the circumference of the circle X at the point W. Join D W. Produce E G to meet and terminate in the circumference of the circle Z, at the point O, and join O P. With P as centre, and P E as interval, describe an arc to meet P F produced at the point *n*, and join E *n*.

My Letter of the 7th inst. was a sort of episode on the things of the day, but the circumstances out of which it arose, so far changed the current of my thoughts, that I was induced to refer to a Letter I had written to the Rev. Geo. B. Gibbons, as far back as the 8th September, 1866, in which I dealt with a geometrical figure, very different, but possessing all the properties of that represented by Diagram XVII., and I am induced to give you some copious extracts from that Letter which commenced as follows:—

" I only found my way home a day or two ago, after attending the late meeting of *The British Association for the Advancement of Science*. During my absence, your favours of the 20th, 21st, and 23rd ult., reached Barkeley House, and have now had my careful attention. I shall, in the first place, notice that of the latter date."

" That Letter you commence by observing :—'*I plainly see that it is useless to prolong our controversy, for we are beginning not to understand each other's speech. You must attach some other meaning than mine to the phrase 'vary as,' otherwise, it is absolutely impossible that you should maintain that the sine varies as the arc.*' It would indeed be *strange*, if our long correspondence should result in finding ourselves incompetent to express our ideas in written language, so as to convey to each other the honest conviction of our minds. This, however, I believe to be '*absolutely impossible.*'

Towards the close of the Letter, you observe:—'*I am quite contented to accept your remark. You are making an attempt to escape from a controversy which you begin to find perplexing. This is true. I do find it perplexing. You have, I confess utterly perplexed me by the strange doctrine that the sine varies as the arc: still more—if possible—by fixing on arcs above* 30°, *whereas, the larger the arc the more it deviates from varying as the sine.*'"

"Now, I would earnestly and respectfully ask you to inspect the enclosed diagram. Without presuming to imagine that '*you are learning from me Geometry and Trigonometry,*' I think I may safely venture to assert, that the geometrical figure represented by the diagram, will be perfectly new to you. It may be true that '*you have studied Mathematics on a rather wide range, far longer than I have,*' (I am upwards of three score, and am, I suspect, your senior in point of age), but, whether or not, I shall *assume* that you are a master in mathematical science. On this assumption I may observe: It will be obvious to you, from a mere inspection of the diagram, or, as you would say '*by sight*', that D E is a diameter of the circle X, D K a side of an inscribed equilateral triangle, D W a side of an inscribed square, and D G a side of an inscribed regular hexagon to the circle X. And similarly, P E is a diameter of the circle Z, P V a side of an inscribed equilateral triangle, P S a side of an inscribed square, and P O a side of an inscribed regular hexagon to the circle Z. Now, my dear Sir, conceive the line D E to revolve round D (and *you know* that this conception is admitted and adopted by trigonometrical writers, in treating of the magnitude of angles), until it coincides with the line D S. In the course of its revolution, the portion of the line D E, within the circle X, will gradually diminish, and when it arrives at the point of coincidence with the line D S, will have reached its vanishing point; that is to say, no part of the line D E will remain within the circle X."

"Again: Conceive a line P E to revolve round P simultaneously with the revolution of the line D E, round D. Then: When D E is coincident with, that is to say, rests on the line D K, the line P E will be coincident with the line P V. When D E is coincident with D W, P E will be coincident with P S. When D E is coincident with D G, P E will be coincident with P O. Now, since the radius of the circle Z, is the double of the radius of the circle X, it follows of necessity, that the lines P V, P S, and P O, are the doubles of the lines D K, D W, and D G. But, D E is the sine of half the arc E S P, and if D E = 1, then, $\frac{1}{2}$(P V) = D K = $\sqrt{\cdot75}$ = ·8660254; and ·8660254 is the sine of half the arc V O P. $\frac{1}{2}$(P S) = D W = $\sqrt{\cdot5}$ = ·7071068; and ·7071068 is the sine of half the arc S O P. And, $\frac{1}{2}$(O P) = G D = $\frac{1}{2}$ = ·5, is the sine of half the arc subtending the chord O P. Thus, $\frac{1}{2}$(P E) is the sine of an angle of 90°. $\frac{1}{2}$(P V) is the sine of an angle of 60°. $\frac{1}{2}$(P S) is the sine of an angle of 45°; and $\frac{1}{2}$(O P) is the sine of an angle of 30°; and it is obvious, that *geometrically* the sine varies as the arc, and is in harmony with the fact which you have distinctly admitted in a previous Letter, namely: '*The sine of an arc is half the chord of twice the arc.*'"

"I have now explained the sense in which I employ the words, '*the sine varies as the arc*,' and I think it is '*absolutely impossible*' that you can any longer have a doubt on your mind as to my meaning."

"Now, my dear Sir, it will be obvious to you as a master in mathematical science, from a mere inspection of the diagram, that the arcs subtending the chord E C and C F are equal to one-sixth part of the circumference of the circles X or Y; and the arc E M F subtending the chord E F equal to one sixth part of the circumference of the circle Z; and since the circumferences of circles are to each other as their radii, it follows of necessity, that the length of the arc E M F is double the length of the arc E C or C F; or in other words, the length of the arc E M F is equal to the sum of the arcs E C and C F. Now, conceive the arc E M F to be a steel spring, and the line D M to denote a thread or wire fixed at M, then, we have only to conceive the thread or wire denoted by D M to be drawn inwards until the point M rests on the point C, to make the arc E M coincident with the arc E C, and the arc M F coincident with the arc C F."

"Again: It will be obvious enough to you that the arcs G D and D H are equal to their opposite arcs E C and C F. Now, conceive the arcs G D and D H to be a steel spring fixed at the points G, D, and H. Then: If the spring be detached at D, it will fly off to its natural position and form the arc G N H, similar and equal to its opposite arc E M F, producing a perfect ellipse or oval composed of four arcs, E K G, G N H H L F, and F M E, tangental to each other. You will find that the periphery of the ellipse or oval is equal to two-third parts of the circumference of the circle Z, or, four-third parts of the circumference of the circles X or Y ; and the longer diameter of the ellipse equal to three times the radius of the circles X or Y, or, three-fourth parts of the diameter of the circle Z ; therefore, K L = A P. Now, conceive the arc E W D, that is, a semi-circumference of the circle X, to be a steel spring fixed at the points E, W, and D. Then, if the spring be set free at W and D, it will fly off in the direction of the arc EVS, and if intercepted at S, will become coincident with the arc EVS, that is, with a quadrant of the circumference of the circle Z. Now, my dear Sir, I suspect it would *perplex you* more than you have ever been *perplexed* by anything I have written, if you attempted to produce, by practical Geometry, any other description of ellipse or oval within the circle Z, of which the periphery and longer diameter shall bear any definite relation to the circumference and diameter of the circle Z."

"Again: The triangles E C D and E F P are similar right-angled triangles; therefore, the angles D and P are equal angles. E F is the sine of the angle P, and the arc E M F subtending the chord E F, is bisected by the line D M, and D M is a radius of the circle Z. Now, E *n* the chord of the arc E *m n*, is also bisected by D M, but D M produced will not bisect the arc E *m n*. But, if from the angle P we draw a straight line, joining P M, the line P M produced will bisect the arc subtending the chord E *n* at the point *m*. Now, because P E = 2 (D E), and because the arcs subtending angles are to each other as the diameters of their respective circles, the arc E M F, subtending the chord E F, is equal to twice the arc E *p* C, subtending the chord E C. But, the chord of half the arc E M F is half the chord of the arc E *m n*; therefore, the arcs E M F and E *m n* are equal, and both equal to the sum of the arcs subtending the chords E C and C F. Thus, if we conceive the arc E *m n* to be a steel spring fixed at E, and the line H *n* to denote a thread or wire attached at *n*, we have only to conceive the thread or wire denoted by the line H *n* to be drawn inwards towards the angle P, until *n* an extremity of the arc E *m n* rests on the angle F, when the arcs E *m n* and E M F will exactly coincide."

"Now, it is self-evident, that the line E F is longer than the line E G, and yet, they are the chords of the equal arcs E M F and E W G; and similarly, E M is a longer line than E C, and yet, they are the chords of the equal arcs E *o* M and E *p* C. But, the arc E W G is a sharper curve than the arc E M F; consequently, the ratio of half the chord to half the arc in the one case, differs from, and bears no arithmetical relation to, the ratio of half the chord to half the arc in the other; and similarly, the arc E *p* C is a sharper curve than the arc E *o* M; consequently, the ratio of half the chord to half the arc in the one case, differs from, and bears no arithmetical relation to, the ratio of half the chord to half the arc in the other. We thus get at the *root of the fallacy* by which you are led to *fancy*, that the THEORY which makes 8 circumferences of a circle exactly equal to 25 diameters, would make the perimeter of a regular polygon of 24 sides greater than the circumference of its circumscribing circle. And hence the utter absurdity of the orthodox *assumption* that the ratio of diameter to circumference in a circle, can be arrived at even approximately, by means of regular polygons, whether inscribed or circumscribed to a circle. Now, my dear Sir, I tell you without the least doubt or hesitation, that it is the Orthodox Mathematician, and not *I*, who bases his starting point in the search after π, upon an *assumption without the shadow of a proof*."

In my Letter to the Rev. George B. Gibbons, I went into the proofs of some other facts in connection with the remarkable geometrical figure, represented by Diagram XVII:, which for my present purpose it is unnecessary to refer to. I then went on to observe:—

"This epistle has already run to too great a length, and I must bring it to a close as quickly as possible. Now, I fancy that I hear you exclaim!—This is all very pretty, but *cui bono?* What is the conclusion at which you arrive from all this? Is π to be "*lugged in somehow?* I will answer these questions, and in doing so, will not speak of, or in any way introduce the symbol π."

"Well, then, if D E the radius of the circle Z = 1, P E the diameter of the circle and hypothenuse of the right-angled triangle E F P = 2. But P E, the hypothenuse of the triangle E F P, is to E F the perpendicular in the ratio of 2 to 1, by construction; therefore, $\frac{PE}{2} = \frac{2}{2} = 1 = EF$, and $\sqrt{PE^2 - EF^2} = \sqrt{2^2 - 1^2} = \sqrt{3} =$

F P the base of the triangle E F P. But, V F and V P = F P, and V F isbisected at A; therefore, F A P and V A P are similar triangles; therefore, V F = $\sqrt{3}$, and F A and V A = $\frac{\sqrt{3}}{2} = \sqrt{\frac{3}{2^2}} = \sqrt{\cdot 75}$; therefore, $\sqrt{(FP^2 - FA^2)} = \sqrt{3 - \cdot 75} = \sqrt{2\cdot 25} = 1\cdot 5 = PA$; and it follows, that P A is equal to three-fourth parts of P E, the diameter of the circle Z; therefore, P E — P A = 2 — 1·5 = ·5 = A E, the semi-radius of the circle Z. But, F P and F A are incommensurable quantities; that is to say, we cannot give exact arithmetical expression to the roots of the symbols $\sqrt{3}$ and $\sqrt{\cdot 75}$; and yet, the fact is incontrovertible, that the values of P A and A E are exact arithmetical quantities, when D E a radius of the circle Z = 1. Thus, we have an example of working from finite quantities, through incommensurable quantities, to finite quantities again. Now, A E is bisected at R by the line K C; therefore $\frac{AE}{2} = \frac{\cdot 5}{2} = \cdot 25$ = A R and E R; therefore, E R is the demi-semi-radius of the circle Z, and we obtain the following identities: $50\,(ER^2) = 12\cdot 5\,(EA^2) = 2\,(ED^2 + RD^2) = PR^2 + ER^2 = (SD^2 + DR^2 + SR^2) = (TD^2 + DR^2 + TR^2)$; and I maintain, that all these identities are equal to the area of the circle Z."

At this point I must call your attention, Mr. Editor, to the following facts. First fact: All the foregoing identities = $3\frac{1}{8}$ times the area of a square on the radius of the circle Z. Second fact: F A is a straight line drawn from the right angle F in the triangle E F P perpendicular to its opposite side E P; therefore, the triangles F A E and F A P are similar triangles, and are similar to the whole triangle E F P. (Euclid: Prop. 8, Book 6). Third fact: From the right angle D in the triangle S D R, draw a straight line perpendicular to its opposite side S R. This line will bisect the angle K D W, and divide the triangle S D R into two similar triangles, both of which will be similar to the whole triangle S D R. (Euclid: Prop. 8: Book 6.) How could these facts be possible, if there were no definite relation between the diameter and circumference of a circle?

My Letter of the 8th September, 1866, to the Rev. Geo. B. Gibbons, concluded as follows:—

"Now, my dear Sir, '*I am quite contented to accept*' your assumption, that you are very much my superior in Mathematical knowledge; and equally contented to take you at your word, when you say: '*you are not learning from me Geometry and Trigonometry.*' You assure me—and not doubting that you express the honest conviction of your mind, '*I am quite contented to accept*' your assurance—that '*I never offer you anything but what you can see in a moment by the simplest Geometry.*' This being the position of affairs, it is perfectly clear, if I am wrong in maintaining all the foregoing identities to be equal to the area of the circle Z, that you can prove it; and—*you*, not I, having sought the controversy—this you are bound to do as a fair and candid controversialist. If you find this to be impossible—as I think you will—your duty becomes plain and simple, and I cannot imagine that you will venture to display a want of candour, by hesitating to admit that I am right."

"I must apologise for the length of this epistle, but, in justice to the subject and to myself, I could not make it shorter."

Not hearing from Mr. Gibbons with his usual punctuality, on the 19th September I addressed the following short Letter to him:—

"I wrote you a long Letter dated the 8th instant, to which I have not received any reply. Will you kindly favour me with answers to the following questions?—Has that Letter from some cause failed to find its way into your hands? If not, do you wish me to accept, as your reason for not replying, the closing paragraph of your last Letter? The paragraph in question runs thus:—'If, therefore, I fail to reply to any Letter you may favour me with, put it down to this—that *I have nothing to say beyond what I have laid before you already.*'"

The reception my Letter met with at the hands of the Rev. Geo. B. Gibbons, may be inferred from the following, which commenced a Letter of mine to that gentleman, dated 29th September, 1866:—

"Your Letter without date, but bearing the Launceston postmark of yesterday, is to hand. You observe:—'It surely cannot require so complicated a diagram as yours of 8th September, to find the equality—if such exists—between a rectilineal and curvilinear area.' Certainly not! And probably you would never have seen that complicated figure, had I been fortunate enough to induce you to grapple with arguments founded upon figures much less complicated. But why refer to that figure at all, when you do not so much as attempt to deal with it. Is it that you find yourself incompetent to the task of grappling with it? If so, pray admit it, and there will be no occasion to '*continue the controversy as long as we both shall live.*' You say:—'*There must be some time at which I may without offence retire from it.*' Certainly! You may retire from it at once, if you can do so *without offence* to your own conscience. So far as I am concerned, there will be no *offence.* On the contrary, I shall ever feel grateful to you for much that you have taught me. Should you retire from the controversy, I shall continue for a time to write you, with a view to publication; but, as geometrical data admit of no doubt, and as logical reasoning upon such data can never lead us into error, I have still a hope that we shall logically reason our differences to a true and just conclusion."

Although our correspondence was continued till the month of May of this year, Mr. Gibbons never again referred to the complicated figure of 8th September, 1866, and I presume is at this moment of opinion, that I am in error when I say, that the periphery of the oval or ellipse is to the circumference of the circle Z, in the ratio of 2 to 3.

Now, there are facts connected with this remarkable geometrical figure which I had not the opportunity of pointing out to Mr. Gibbons, and to some of these facts, Mr. Editor, I shall now direct your attention.

Well, then, the fact is incontrovertible, that 3·125 times the area of a square on the radius of the circle Z is equal to the identities* I have deduced from the geometrical figure represented by the diagram, and on the theory that 8 circumferences of a circle are exactly equal to 25 diameters, makes $\frac{25}{8} = 3{\cdot}125$ the arithmetical value of π.

Now, $\frac{\pi}{4}$ = area of a circle of diameter unity, whatever be the value of π. But, $\frac{3{\cdot}125}{4} = {\cdot}78125$, and $\cdot78125 : 1 :: 1 : 1{\cdot}28$; therefore, 1 is the mean proportional between ·78125 and 1·28. Now, 5 (E R) = S R or T R, and is equal to five-sixth parts of the longer diameter of the ellipse. But, 5 (E R) = (D E + E R), and D E + E R = 5, when D E = 4. Then: If with D as centre and (D E + E R) as interval, we describe a circle, $3·125\ (5^2) = 78·125$ = area of the circle; and $4\ (5^2) = 100$ = area of a circumscribing square to the circle; and 78·125 : 100 : : 100 : 128; therefore, 100 is the mean proportional between 78·125 and 128. Hence: 128 is the area of a circumscribing square to a circle of which the area is 100. Now, 100 : 128 : : 128 : 163·84, and $\frac{128}{·78125} = 163·84$; therefore, 163·84 is the area of a circumscribing square to a circle of which the area is 128. For, $\frac{1}{2}(\sqrt{163·84}) = \sqrt{40·96}$ = radius of the circle, and πr^2 = area in every circle; therefore, $3·125 \times 40·96 = 128$ = area of the circle. Again: 128 : 163·84 : : 163·84 : 209·7152, and $\frac{163·84}{·78125} = 209·7152$; therefore, 209·7152 is the area of a circumscribing square to a circle of which the area is 163·84, and so on we might proceed, *ad infinitum.*

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* In the original Letter, I employed the word equations, and in this have substituted the word identities. This I have done out of deference to the opinion of the Rev. Professor Whitworth, who says that identities is the word that would be adopted by Mathematicians.

The question then arises, Can these facts be connected with any other figure in the diagram? I answer, Yes! E P is a diameter of the circle Z, and S P is a side of an inscribed square to the circle Z, and E F P is a right-angled triangle of which the hypothenuse and perpendicular are in the ratio of 2 to 1. Hence: $(EF^2 + FP^2 + EP^2) = 128$, when E D the radius of the circle Z = 4, and is equal to the area of a circumscribing square to a circle, of which S P is the radius. And, $(EC^2 + CD^2 + ED^2)$, that is, the sum of the squares of the sides of the right-angled triangle E C D = 32, and is equal to the area of a square of which S P is a side; that is, = the area of an inscribed square to the circle Z.

How, my good Sir, could such facts exist, if there were no definite relation between the diameter and circumference of a circle? How could such facts exist, if the area of a circle of diameter unity were arithmetically inexpressible. Well, then, Mr. Editor, I ask not your opinion, but I tell you, however much you may be pleased to despise them, that these facts demonstrate—beyond the possibility of dispute or cavil by any *honest* Geometer and Mathematician—the truth of the THEORY that 8 circumferences = 25 diameters in every circle, and makes $\frac{1}{3\cdot125}$ the true arithmetical expression of the ratio between the diameter and circumference in every circle.

The foregoing facts are very far from exhausting the properties of this remarkable geometrical figure, in proof of which I will mention one, which you may readily verify. $3\,(SP^2) = 4\,(KR^2 + RD^2 + KD^2)$; that is, 3 times the area of an inscribed square to the circle Z = 4 times the sum of the squares of the sides of the right-angled triangle K R D, and this equation = area of a regular inscribed dodecagon, to a circle of which S P is the radius.

Enough, if you—Mr. Hepworth Dixon—be an honest scientific Journalist; and if not, I shall have nothing to retract.

I am, Sir,
Yours respectfully,
JAMES SMITH.

The two last Letters were not directed to the Editor of the *Athenæum*, but to William Hepworth Dixon, Esq., Editor of the *Athenæum*. That *unscrupulous sophist*, Mr. A. De Morgan, would have me believe that Mr. Hepworth Dixon did not present him with my Letter of the 7th November, or with any of my Letters subsequent to that of the 15th November, 1867. Now, the reader will observe, that in my Letters of the 7th and 15th November, I have introduced diagrams shewing "*an ellipse or oval.*" In the article: Pseudomath, Philomath, and Graphomath, Mr. A. De Morgan says:—"*Further thanks for Mr. Smith's Letters to you of* Oct. 15, 18, 19, 28, *and* Nov. 4 *and* 15. *The last of these letters has two curious discoveries.*" Passing by the first as quite immaterial to the question at issue, he observes:—"Secondly: An '*ellipse or oval*' is composed of four arcs of circles. Mr. Smith has got hold of the construction I was taught, when a boy, for a pretty four-arc oval. But my teachers knew better than to call it an ellipse: Mr. Smith does not; but he produces from it such confirmation of $3\frac{1}{8}$ as would convince any honest Editor."

Now, referring to the diagram enclosed in my Letter to the Editor of the *Athenæum* of the 15th November, 1867 (see *Diagram XVII.)*, which Mr. A. De Morgan admits came into his hands; it will be self-evident to the reader, that all the sides of the rectangle E F H G, are not equal; and yet, that all the sides are subtended by equal arcs; and it follows of necessity, that the curvilinear figure about the rectangle is a perfect "*ellipse or oval*," whatever that "*unscrupulous sophist*," Mr. A. De Morgan, may be pleased to say to the contrary. In my correspondence with the Rev. George B. Gibbons, by means of a very different geometrical figure (See Diagram VI.), I proved that *equal* arcs may be subtended by *unequal* chords, which drew from Mr. Gibbons the following remarks:—"*Yours just received, asserts that unequal chords may be subtended by equal arcs. So they may, but then the arcs are not arcs of a circle. It is true of an ellipse or parabola, but not true of a circle.*" (See my correspondence with the Rev. Professor Whitworth, page 124).

Let Professor A. de Morgan answer the following question: Which of the three, the Rev. George B. Gibbons, the Rev. Professor Whitworth, or Mr. A. De Morgan, is "*the what's-his-name that rushes in where thing-em-bobs fear to tread?*" In putting the question in this form, I take my cue from the Professor's own words, which may be found among his *Budgets of Paradoxes.*

BARKELEY HOUSE, SEAFORTH,
16th December, 1867.

TO THE EDITOR OF THE ATHENÆUM.

SIR,

The question has sometimes occurred to me:—Can—in the opinion of a Mathematician—any good thing come out of my native town? It is gratifying to find, that to this question, the Editorial "*we*" *of the Athenæum*, or, in other words, the "*King of English Critics*," can answer emphatically "Yes." The circumstances out of which this answer to the question arose, reminds me responding to the natural number ·3162277 is 9·4999998, and this is the nearest Log.-Sin. to an angle of 18° 26′, and only differs from Hutton's Tables by ·0000367. But mark! By Tables, the difference between the Log.-Cos. of an angle of 18° 26′ and an angle of 18° 27′ is 421; and the difference between the Log.-Sin. of an angle of 18° 26′ and an angle of 18° 27′ is 3788. But mark further: The *mantissa* of the Log.-Sin of the angle F D G, is as nearly equal to ·5, the sine of an angle of 30°, '*as the number of decimals employed will give it.*' Now, when we make the sides of the triangle O D E, ·3, ·4, and ·5, and O D the radius of the circle, then, 6 (O D) = the perimeter of a regular inscribed hexagon to a circle of diameter unity; therefore, O D = half the chord subtended by an angle of 60° to a circle of radius 1. Has not this something to do with the solution of a right-angled triangle?"

This brings me to one of the most amusing incidents in my encounter with the "*dragon*" of St. John's.

I had impugned the accuracy of our Mathematical Tables, and for this reason, he had charged me with childishness, ignorance, and dishonesty. In his Letter to me of the 23rd January, having proved that the Sin. of half the angle O D E, that is, the Sin. of the angle F D G, = ·3162277...he said: "I test my work thus:—

Sin. 2ϕ = ·6000000, by hypothesis.

Sin. 36° 52′ = $\frac{\cdot 5999549}{\cdot 0000451}$: difference by Tables 2326: so that, 2ϕ = 36° 52′ $\frac{451}{2356}$ nearly.

"Again:

"Sin. ϕ = ·3162277.

"Sin. 18° 26′ = $\frac{\cdot 3162010}{\cdot 0000267}$: difference by Tables 2760; so that, ϕ = 18° 26′ $\frac{267}{2760}$ nearly: agreeing as nearly as the number of decimals employed will give it."

My reply to this ran thus:—"Now, Sir, I put a question, and appeal to your common sense for an answer: Is it conceivable that you could have advanced an argument better calculated to prove your *assumption* of the infallibility of Mathematical Tables to 7 places of decimals?"

I Professor de Morgan had caught me so tripping, might he not have truly said :

"If you want to establish a point,
Begin by assuming it true ;
For if it is wrong it's no matter,
And if it is right it will do."

I had brought you to this stage of my encounter with the dragon of St. John's, Mr. Editor, when some six weeks ago I was diverted from my course by "*the things of the day.*" Having now picked up the thread of my story, I may take a fresh departure. Well, then, my Letter of the 2nd March to the Rev. George B. Gibbons concluded with a promise to show him that he was wrong in making the assertion that I don't understand the construction and use of Mathematical Tables. The following is a copy of my next Letter to that gentlemen :—

BARKELEY HOUSE, SEAFORTH,
5th March, 1867.

MY DEAR SIR,

"It only remains for me to fulfil the promise made at the close of my last Letter, and then, if it be your pleasure we may close our correspondence."

"I must repeat the passage from your Letter of the 20th February, already quoted in mine of the 2nd inst. "*And then your saying I charge you with ignorance or dishonesty--simply because you are not familar with the construction of a certain set of tables! A man may be ignorant of one particular thing without being generally ignorant, and that you don't understand Hutton's Tables is made still clearer by your Letter just received.*" Now, my dear Sir, there can be no mistake as to the fact, that in this quotation you charge me with ignorance with regard to the construction of Hutton's Mathematical Tables. From this charge I must *purge myself*, by furnishing you with clear and distinct evidence, that I am not only familiar with the construction, but that I also know how to make a proper use of these Tables. Having done so, and thus put "*you in the wrong*," if you choose to be silent, I must be content. If you answer by dogmatical assertion, without a shadow of proof, and I reply in a tone of sarcastic severity, "*you must bear it;*" but you will have yourself alone to blame. If I am wrong with reference to what I am about to bring under your notice, as a Mathematician, *you can prove it by logical reasoning;* and if you decline to grapple with my proofs, I shall simply draw the inference, that you have resolved to make your stand as the champion of geometrical and mathematical absurdity, and not as what till lately I have believed you to be, a sincere and earnest enquirer after scientific truth.

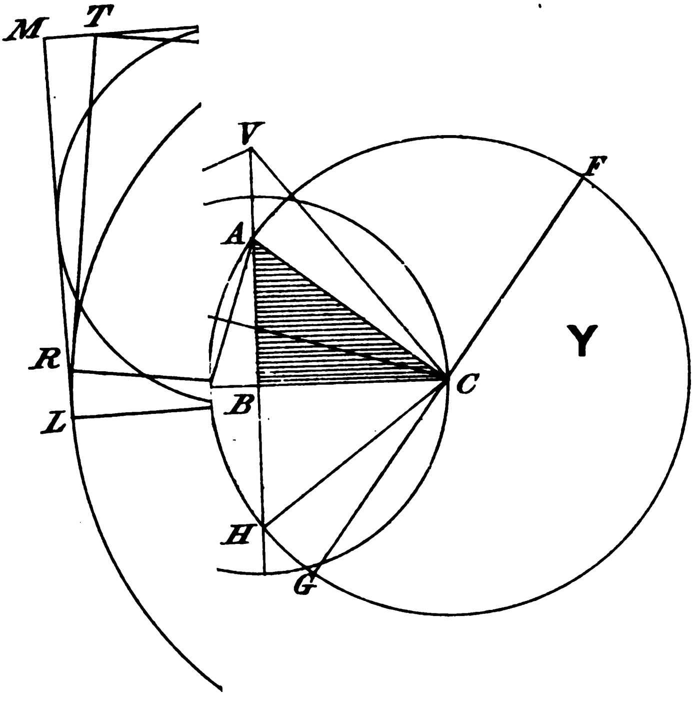
The enclosed diagram is a fac-simile of that contained in my Letter of the 11th February, with certain additions and omissions. *(See Diagrams V.* and *XVIII.)*

Now, referring to Diagram XVIII., because V B is perpendicular to O C, V B O, and V C B are right-angled triangles, and V B is the sine of, and common to, the two angles V O B and V C B. Again: Because V B is equal to A C, and A C : B C : : 5 : 4, by construction, the sides of the right-angled triangle V B O are in the ratio or proportion of 5, 12, and 13 exactly; that is to say, V B : B O : : 5 : 12; and, V B : V O : : 5 : 13, by construction. Hence: V B O is a commensurable right-angled triangle, derived from the consecutive numbers 2 and 3; that is to say, A V = the difference between A C and A B, and when A B = 3, A B + A V = 3 + 2 = 5 = A C. Twice the product of A B and A V = 2 (3 × 2) = 12 = B O; therefore, $\sqrt{VB^2 + BO^2} = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} =$ B O + 1 = 13 = V O. But, V B = A C, by construction; and if A C = 5, B C = 4, by construction; therefore, $\sqrt{VB^2 + BC^2} = \sqrt{5^2 \times 4^2} = \sqrt{25 + 16} = \sqrt{41} = 6{\cdot}403125......$ = V C, the hypothenuse of the right-angled triangle V B C, and the triangle V B C is incommensurable. It is obvious, from a mere inspection of the diagram, that the two triangles V B O and V B C together form the oblique-angled triangle O V C.

Now, let the angle V O C be an angle of 22° 37, O V C an angle of 106° 3', and the length of O V the hypothenuse of the right-angled triangle V B O = 130 miles. What is the length of the side V C in the oblique-angled triangle O V C?

Then: 180° — (angle V O B + angle O V C) = 180° — (22° 37' + 106° 3') = (180° — 128° 40') = 51° 20'; therefore, the angle V C B is an angle of 51° 20'.

Now, in the right-angled triangle V B C, if V B = 5, B C = 4, and $\sqrt{VB^2 + BC^2} = \sqrt{5^2 + 4^2} = \sqrt{25 + 16} = \sqrt{41} = 6{\cdot}403124...$


M
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= V C, the hypothenuse of the right-angled triangle V B C; therefore, $\frac{VB}{VC} = \frac{5}{6{\cdot}403124} = {\cdot}7808688$, is the arithmetical value of the sine of the angle V C B. In the right-angled triangle V B O; if V B = 5, V O = 13, and V B is the sine of the angle V O B; therefore, $\frac{VB}{VO} = \frac{5}{13} = {\cdot}3846153$, (·384615 is a recurring decimal), is the arithmetical value of the sine of the angle V O B. The Logarithm of the natural number ·7808688 is 9·8925781, and this is the Log.-sin. of the angle V C B. The Logarithm of the natural number ·3846153 is 9·5850266, and this is the Log.-sin of the angle V O B.

Hence:

As Sin. of angle V C B = Sin. 51° 20′ ....Log. 9·8925781
: V O the given side of the triangle O V C =
130 miles ....Log. 2·1139434
: : Sin. of angle V O B = Sin. 22° 37′....................Log. 9·5850266

11·6989700
9·8925781

: V C the required side of the triangle O V C
= 64·03124 miles...................................Log. 1·8063919

By Hutton's Tables:

As Sin. of angle V C B = Sin. 51° 20′ ....Log. 9·8925635
: V O the given side of the triangle O V C =
130 miles ... .Log. 2·1139434
: : Sin. of angle V O B = Sin. 22° 37′ ....Log. 9·5849685

11·6989119
9·8925635

: V C the required side of the triangle O V C
= 64·029 miles ....Log. 1·8063754

This makes V C less than its known and indisputable value, and proves beyond the possibility of dispute or cavil, that Hutton's Tables are erroneous.

Again: Let the angle V C B in the right-angled triangle V B C be an angle of 51° 20′, and the length of the side V B be 5 miles. What is the length of the side B C?

Then: If V B = 5, B C = 4, by construction; and B is a right angle; therefore, (90° — Sin. of angle V C B) = (90° — 51° 20′) = 38° 40′; therefore, B V C is an angle of 38° 40′. But, B C is the sine of the angle B V C; therefore, $\frac{BC}{VC} = \frac{4}{6{\cdot}403124} = {\cdot}6246950$, is the arithmetical value of the sine of the angle B V C. The Logarithm of the natural number ·6246950 is 9·7956680, and this is the Log.-sin. of the angle B V C.

Hence:

As Sin. of angle V C B = Sin. 51° 20′ ....Log. 9·8925781
: V B the given side of the triangle V B C =
5 miles ....Log. 0·6989700
: : Sin. of angle B V C = Sin. 38° 40′ ....Log. 9·7956680

10·4946380
9·8925781

: B C the required side of the triangle V B C
= 4 miles...Log. 0·6020599

Thus, by a right application of true Logarithms, we arrive at the exact value of the length of the line B C; and we should obtain the same result whether we call V B 50 miles, or 50,000 miles; the only difference would be a change in the index of the Logarithms.

By Hutton's Tables:

As Sin. of angle V C B = Sin. 51° 20′...............Log. 9·8925365
: V B the given side of the triangle V B C = 5 miles..Log. 0·6989700
: : Sin. of angle B V C = Sin. 38° 40′ ....Log. 9·7857330

10·4947030
9·8925365

: B C the required side of the triangle V B C ......Log. 0·6021665

And 0·6021665 is the Log. of 4·001 ..miles.

Now, V B is to B C in the ratio or proportion of 5 to 4, by construction; and the relations of side to side by construction can never be over-ridden by Logarithms, which are only approximations. Now, if V B = 5, B C = 4, neither more nor less, by construction. But, worked out by Hutton's Tables, B C is made greater than 4, which is impossible, and again demonstrates that Hutton's Tables are erroneous, beyond the possibility of dispute or cavil.

Again: Let the angle V C B in the right-angled triangle V B C be an angle of 51° 20′, and the length of the side V B be 5 miles, what is the length of the side V C? Then: B is a right angle = 90°.

Hence:

As Sin. of angle V C B = Sin. 51° 20′ ....Log. 9·8925781
: V B the given side of the triangle V B C = 5
miles ....Log. 0·6989700
: : Sin. of angle B = Sin. 90° ....Log. 10·0000000

10·6989700
9·8925791

: V C the required side of the triangle V B C
= 6·403124 miles ....Log. 0·8063909

Well, then, 0·8063909 is the Logarithm corresponding to the natural number 6·403124, true to 4 places of decimals, and is therefore the Logarithm of $\sqrt{VB^2 + \overline{BC}^2} = 6{\cdot}403124\ldots$, which is the known and indisputable value of the side V C in the triangles V B C and O V C, when V B = 5.

By Hutton's Tables:

As Sin. of angle V C B = Sin. 51° 20′ ....Log. 9·8925365
: V B the given side of the triangle VBC = 5 miles Log. 0·6989700
: : Sin. of angle B = Sin. 90° ....Log. 10·0000000

10·6989700
9·8925365

: V C the required side of the triangle V B C......Log. 0·8064335

Now, 0·8064335 is the Logarithm corresponding to the natural number 6·4037..., which makes V C greater that its known and indisputable value, and again proves in a way that no candid Mathematician will venture to dispute, that Hutton's Tables are erroneous.

Now, my dear Sir, I have not only proved that Hutton's Tables are slightly in error with reference to the sines, cosines, &c., of angles, but I have done more. I have proved that in a right-angled triangle, the values of the sides when worked out and ascertained by true Logarithms, are in perfect harmony with the values as given by construction, and I cannot help thinking a moment's reflection will convince *you*, that this must necessarily be so, and proves that I am right.

Now, the figure A K H G C is a quadrant of the circle Y; therefore, the angle A C G is a right angle = 90°, and is divided into three angles A C B, B C H, and H C G. These angles correspond with the angles M O L, L O F, and F O E in the geometrical figure represented by the diagram enclosed in my Letter of the 2nd inst. *(See diagram enclosed in my Letter to you of 4th November).* Some most interesting proofs of the truth of the THEORY that 8 circumferences = 25 diameters in every circle arise out of these facts, but this Letter has run on to such a length that I must bring it to a close, and reserve any further observations for another opportunity.

Believe me, my deár Sir,
Very truly yours,
JAMES SMITH.

THE REV. GEO. B. GIBBONS, B.A.

I hope to give you a copy of my next Letter to the "*dragon*" of St. John's, when I again address you.

I am, Sir,
Yours respectfully,
JAMES SMITH.

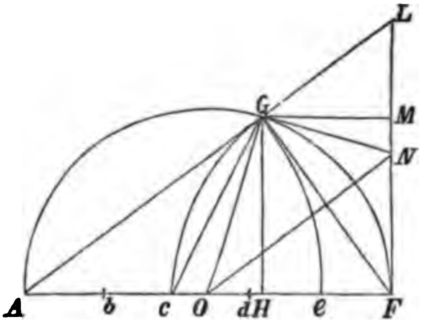
If we assume $\pi = 3$, or $\pi = 4$, or $\pi =$ any finite and determinate arithmetical quantity intermediate between 3 and 4, we get the equation or identity, $4\left(\frac{\pi}{2}\right)^2 = \pi^2$. But, it is self-evident that π must be greater than 3 and less than 4, since 3 = the perimeter of an inscribed regular hexagon, and 4 = the perimeter of a circumscribing square, to a circle of diameter unity; and it follows, that the value of the symbol π *cannot be an indeterminate arithmetical quantity.* Hence: If we assume $\pi = 3{\cdot}1$, or $\pi =$ the integer 3 with any number of decimals, the equation or identity, $4\left(\frac{\pi}{2}\right)^2 = \pi^2$, holds good. *How, then, is it possible, that π can be an indeterminate arithmetical quantity?*

J. S.

APPENDIX C.

In the *Athenæum* of July 25, 1868, there was a notice of my pamphlet "*Euclid at Fault*," in the Article: "*Our Library Table.*" This led me to address a Letter to the Editor, in which I quoted copiously from, and commented freely upon, the Review of Mr. J. M. Wilson's *Treatise on Elementary Geometry*, which appeared in the *Athenæum* of July 18, 1868. Under the impression that Mr. Wilson would be amused, and even pleased with my communication to Mr. Editor—since I defended him from some of the very unfair attacks of the reviewer—I wrote to him and gave him a copy of that communication, which led to an exchange of several Letters between us.

In one of these Letters, dated 1st Aug., 1868, I enclosed, and dealt with, the geometrical figure represented by the diagram in the margin, of which the following may be taken as the method of construction:—



From the point A draw a straight line of indefinite length, and mark off five equal parts, A *b*, *b c*, *c d*, *d e*, and *e* F. Bisect A F at O, and with O as centre and O A or O F as interval, describe the semi-circle A G F. With A as centre, and A*e* = $\frac{4}{5}$(A F) as interval, describe the arc *e* G, and with F as centre and F *c* = $\frac{3}{5}$ (A F) as interval, describe the arc C G, and join A G and F G. Let fall the perpendicular G H. From the point F draw a straight line parallel to G H, and therefore tangental to the semicircle, to meet the line A G produced at the point L. From the point O draw a straight line O N parallel to A L, and join G N and G O. From the point G draw a straight line G M, parallel to O F.

Mr. Wilson returned this Letter and the diagram, and with them the following laconic note :—

DEAR SIR,

At the place marked, in your Letter, the same assumption is again made. I have been under a mistake in corresponding with you, and I should prefer that the correspondence now closed.

Very truly yours,

J. M. WILSON.

The following is a literal quotation from my Letter of Aug. 1, 1868 :—

"Now, making A F = 8, by hypothesis; we can find the values of A *c*, A H, *c* H, H F, A G, G F, G H, and G *c*.

$$AG = \tfrac{4}{5}(AF) = 6{\cdot}4$$
$$GF = \tfrac{3}{5}(AF) = 4{\cdot}8$$
$$AH = \tfrac{4}{5}(AG) = 5{\cdot}12$$
$$GH = \tfrac{3}{5}(AG) = 3{\cdot}84$$
$$HF = AF - AH = 2{\cdot}88$$
$$cH = cF - HF = 1{\cdot}92$$
$$Ac = AH - CH = 3{\cdot}2.$$

Hence:

$\frac{3}{5}$ (AG) = $\sqrt{(AH \times HF)} = \sqrt{14{\cdot}7456} = 3{\cdot}84$; that is, $\frac{3}{5}$ (AG) = $\sqrt{(5{\cdot}12 \times 2{\cdot}88)} = \sqrt{14{\cdot}7456} = 3{\cdot}84$ = GH; and it follows, that $\{Gc^2 + Ac^2 + 2(Ac \times cH)\} = AG^2$; and GHF and GHA on each side of GH are similar triangles, and similar to the whole triangle HGF.

The mistake into which Mr. Wilson stumbled was, in not discovering that the triangle KBM and the triangle OBT in the diagram in "*Euclid at Fault*," (*See Diagram XIV.*,) are not similar right-angled triangles, and consequently would have it that AG = $\sqrt{40}$.

I did not think it necessary to direct Mr. Wilson's attention to the fact, that the sum of the squares of the four sides of the parallelogram GHFM = $2(GF^2)$. I thought this fact could not fail to occur to him.

JAMES SMITH *to* J. M. WILSON, ESQ.

BARKELEY HOUSE, SEAFORTH,
26th October, 1868.

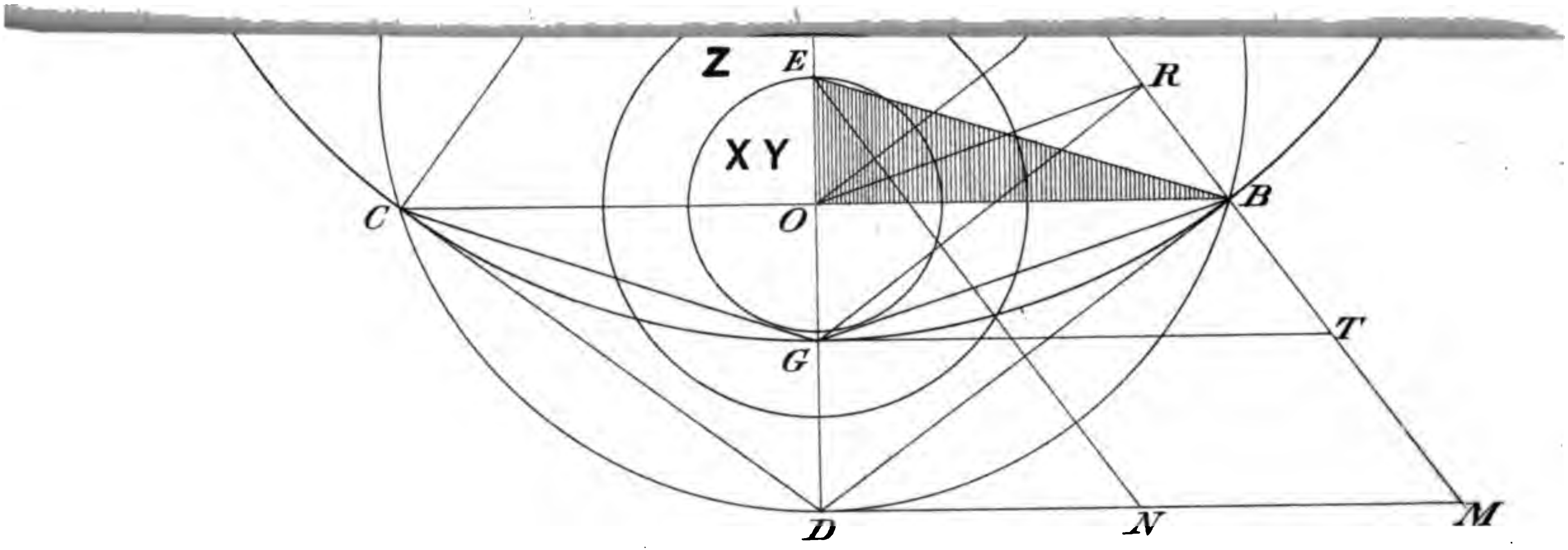
MY DEAR SIR,

When I last addressed you under the date of 4th August I had not seen your work on "*Elementary Geometry*." I have since obtained a copy, and have gone through it very carefully, and can honestly and sincerely assure you—notwithstanding the treatment you have met with from the Editor of the *Athenæum* (or I should rather say, from Professor de Morgan, the Mathematical critic of that periodical)—that I thank you for the publication of such a work. As a text-book on Geometry, it is infinitely superior to Euclid, and I say this without the slighest hesitation. To *me*, it has *suggested* many things, and I cannot resist the temptation to bring some of the facts it has *forced* upon my attention under your notice.

I construct the enclosed geometrical figure in the following way *(See Diagram XIX.)*:—

I draw two straight lines of indefinite length at right angles, making O the right angle. From the point O in the *direction* of O A I mark off four equal parts together equal to O A, and from the point O I mark off in the *direction* of O B three of such equal parts together equal to O B, and join A B, and so, construct the right-angled triangle A O B. It is obvious, or, self-evident, that the sides containing the right angle in the triangle A O B are in the ratio of 4 to 3, by construction. With A as centre and A B as interval, I describe the circle X. I then produce A O and B O to meet the circumference of the circle X at the points G and C, and join A C, B G, and C G, and so, construct the quadrilateral A C G B. I next draw straight lines from the points B and C at right angles to A B and A C, and therefore tangental to the circle X, to meet A G produced at the point D, and so, construct the quadrilateral A C D B. It is obvious, or, self-evident, that the quadrilateral A C G B is wholly within the circle X, and the quadrilateral A C D B partly within, and partly without the circle X. I then bisect A G at the point F, and with O as centre and O F as interval, describe the circle Z, and thus get O F the line that joins the middle points of the diagonals in the quadrilateral A C G B. I next bisect A D at the point E, and with O as centre and O E as interval, describe the circle X Y, and thus get O E the line that joins the middle points of the diagonals in the quadrilateral A C D B. With E as centre, and E A or E D as interval, I describe the circle Y, and join E B. It is obvious, or, self-evident, that E B = E D, for they are radii of the circle Y. I then draw a straight line from the point D at right angles to A D, and therefore tangental to the circle Y, to meet A B produced at the point M, and so, construct the right-angled triangle A D M. I next draw a straight line from the point E the centre of the circle Y, parallel to A M, to meet the base of the right-angled triangle A D M at the point N. It is axiomatic, if not self-evident, that D M is bisected at the point N; for, since A D is bisected at E, and E N parallel to A M, by construction, it follows of necessity, that the line D M is bisected at N. From the angle O, I draw O P perpendicular to A B, and O R parallel to G B; and from the point G, I draw a straight line per-

DIAGRAM XIX.



pendicular to A D, and therefore tangental to the circle X, to meet A M, the hypothenuse of the right-angled triangle A D M, at the point T, and join G R.

Now, it is not necessary to point out to *you*, my dear Sir, the number of right-angled triangles in this geometrical figure, in which the sides that contain the right angle are in the ratio of 3 to 4: it is sufficient to direct your attention to the fact, that A B D, A C D, and A G T are three of such triangles, and that in all of them $3\frac{1}{8}$ times the square of the middle side is equal to the sum of the squares of the three sides. I need not tell *you*, for you know it as well as I do, that "*in any quadrilateral the sum of the squares of the four sides is equal to the sum of the squares of the diagonals, together with four times the square of the line joining the points of bisection of the diagonals.*"* (Miscellaneous Theorems and Problems, 75 and 76. Elementary Geometry, by J. M. Wilson.)

Now, A B D and A G T in the enclosed figure are similar and equal right-angled triangles, but they are widely different in *direction*, and no "*quantity of turning*" could make them coincide; and yet, it is just as conceivable that by *superposition*, the one might be laid on the other so that all the sides and angles should coincide, as that the triangles A B D and A C D, which have a common hypothenuse, and are indisputably similar and equal right-angled triangles, might one be laid upon the other so that all the sides and angles should coincide.

Hence:

The triangles A O B, A O C, and A R G are similar and equal right-angled triangles.

A O : O B :: O B : O D.
A O : O C :: O C : O D.
A B : B D :: B D : B M.
A R : R G :: R G : R T.
A P : P O :: P O : P B.
E N : G D :: E D : O G.

---

*It is self-evident, that C B is a diagonal of, and common to, the quadrilaterals A C G B and A C D B; and it is axiomatic, if not self-evident, that the sum of the squares of the diagonals, together with the area of a circumscribing square to the circle Z, is equal to the sum of the squares of the four sides of the quadrilateral A C G B. And, it is also axiomatic, if not self-evident, that the sum of the squares of the diagonals, together with the area of a circumscribing square to the circle X Y, is equal to the sum of the squares of the four sides of the quadrilateral A C D B.

$$\{BE^2 + EA^2 + 2(EA \times EO)\} = AB^2.$$
$$\{BG^2 + GD^2 + 2(GD \times GO)\} = BD^2.$$
$$\{OR^2 + RB^2 + 2(RB \times RP)\} = OB^2.$$
$$\{GB^2 + BT^2 + 2(BT \times BR)\} = GT^2.$$
$$BD = GT, \text{ and, } RG = OB.$$

All these facts may be demonstrated by "*wielding that indispensable instrument of science, Arithmetic,*" (W. D. Cooley), and when A D = 8, A B = 6·4; B M = 3·6; A M = 10, D M = 6, D N = 3, T M = 2, B T = G D = 1·6; R B = O G = 1·28. R P = 1·024; P B = 2·304; A P = 4·096; A R = 5·12; R T = 2·88; A T = A D = 8; O P = 2·4; O B = R G = 3·84, and G T = B D = 4·8.

The right-angled triangle B O E in the enclosed figure is similar to the right-angled triangle H P T in the diagram in "*Euclid at Fault,*" (*see Diagram XIV.*) and the sides that contain the right angle in both these triangles are in the ratio of 24 to 7. With regard to the former, you will perceive that the ratio of side to side can readily be demonstrated by simple arithmetic, and with reference to the latter, the ratio of side to side can be demonstrated by Logarithms. You will find that when K B the diameter of the circle in Diagram XIV. = 8, that H P = 3, P T = ·875, H T = 3·125, B T = H P = 3. Now, H P B is a right-angled triangle, and H T B a part of it, is an oblique-angled triangle, and according to Euclid, Prop. 12: Book 2, $\{HT^2 + TB^2 + 2(TB \times TP)\} = HB^2$; that is, $\{3{\cdot}125^2 + 3^2 + 2(3 \times {\cdot}875)\}$, or, $\{9{\cdot}765625 + 9 + 5{\cdot}25\} = 24{\cdot}015625 = HB^2$, and is greater than 24. It is admitted that K F × F B = HF^2, that is, 5 × 3 = 15, and it follows that $HF = \sqrt{15}$; and it is self-evident, that H F = P B. Now, $\sqrt{15} = 3{\cdot}873$ nearly, and is less than 3·875, and by no extension of the decimals could we make it approximate nearer to 3·875. To whatever extent we might carry the decimals it would still be less than 3·875 by a distinctly appreciable quantity. Mark the absurdity of the following quotation, which I take from a Mathematical work, in an article on the Quadrature of the Circle. "*It is observed by Montucla that if we suppose a circle whose diameter is a thousand million times the distance of the sun from the earth, the approximate measure of the circumference, as computed by De Lagny, would differ from the true measure by a length less than the thousand-*

*millionth part of the thickness of a hair.** To resolve this into *common sense* we must prove that $\sqrt{15} = 3{\cdot}8749$ with 9 to infinity.

Now, may I ask you, my dear Sir, to take the diagram in "*Euclid at Fault*," (*See Diagram XIV.*) and from T M cut off a part, say T x, making T x equal to T B, and join K x? The line K x would cut the circumference of the circle at a point, say y, nearer to T than H. Then: O B T and K B x would be similar right-angled triangles; and if K B = 8, then, K y would = 6·4, $y x$ would = 3·6, making B x = 6 and K x = 10. Hence: Under no circumstances can we construct a geometrical figure which shall contain a series of right-angled triangles crossing each other in various directions, but when the sides that contain the right angle in the generating triangle of the figure are in the ratio of 3 to 4. The *direction* of K M is different from the *direction* of K x and O T, and the measure of this difference of *direction*, may be taken as the "*quantity of turning*" necessary to turn the heads and systems of such men as Professor de Morgan topsy-turvy.

You know, my dear Sir, that a square exactly equivalent to a given circle exists, but that Mathematicians have not hitherto been able to discover it. A square equivalent to the circle X in the enclosed geometrical figure *(see Diagram XIX.)* can readily be found, and may be isolated and exhibited. From G T cut off a part, say G X, making G X equal to G D. Or, with G as centre and G D as interval, describe an arc, cutting the line G T at X. Produce G A to a point Y, making G Y equal to 2 (A G)— G D, and join Y X. The square on Y X will be the required square.

When I saw your Letter in the *Athenæum*, I was a little surprised that the Editor had inserted it,—I had once to pay very dearly for the insertion of a Letter—but the reason became plain enough to me, when I saw the ungenerous use made of it in the following number of that Journal. Your second Letter, the receipt of which was acknowledged in notices to Correspondents, has of course not been inserted.

I remain, my dear Sir,
Yours very respectfully,
JAMES SMITH.

J. M. WILSON, ESQ.

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* See, Mr. J. Radford Young's Treatise on Euclid's Elements, in *Orr's Circle of the Sciences.*

JAMES M. WILSON *to* JAMES SMITH.

RUGBY, *October* 27, 1868.

DEAR SIR,

Your communication is based on an error: but I decline reading it, or corresponding about it. With thanks for the courtesy of it,

I remain, yours,
J. M. WILSON.

JAMES SMITH *to* J. M. WILSON, ESQ.

BARKELEY HOUSE, SEAFORTH,
27th October, 1868.

MY DEAR SIR,

I have been in correspondence with many Mathematicians during the past eight or nine years, and one of my recent correspondents—a Mathematician of no mean reputation—broadly asserts that "*practical Geometry can prove nothing*," and that to prove that π is not an indeterminate arithmetical quantity, I must demonstrate it by "*a priori or abstract reasoning, that is, without any reference to its arithmetical value*." Is not this equivalent to telling me, that I must find the arithmetical value of π without "*wielding that indispensable instrument of science, Arithmetic?*"

With reference to the geometrical figure represented by the diagram in "*Euclid at Fault*," (*see Diagram XIV.*) there are many points on which you and I are, and must necessarily be agreed. To some of these I am about to refer, and then direct your attention to a very extraordinary anomaly in Mathematics.

When the diameter of the circle = 8, it may be admitted, for the sake of argument, that $KF = \cdot 5$, $FB = 3$, $FH = \sqrt{15}$, $KH = \sqrt{40}$, and $HB = \sqrt{24}$. Hence: By analogy or proportion, $KH : HB :: HB : HM$, that is, $\sqrt{40} : \sqrt{24} :: \sqrt{24} : \sqrt{14 \cdot 4}$; therefore, $HM = \sqrt{14 \cdot 4}$. But, $(HB^2 + HM^2) = BM^2$; that is, $(\sqrt{24}^2 + \sqrt{14 \cdot 4}^2) = (24 + 14 \cdot 4) = 38 \cdot 4 = BM^2$; therefore, $BM = \sqrt{38 \cdot 4}$: or, $KF : FH :: KB : BM$;

that is, $5 : \sqrt{15} :: 8 : \sqrt{38{\cdot}4}$, making $BM = \sqrt{38{\cdot}4}$. $KF : FH : HP : PM$; that is, $5 : \sqrt{15} :: 3 : \sqrt{5{\cdot}4}$; therefore, $PM = \sqrt{5{\cdot}4}$; or, $(HM^2 - HP^2) = (\sqrt{14{\cdot}4}^{\,2} - 3^2) = (14{\cdot}4 - 9) = 5{\cdot}4 = PM^2$; therefore, $PM = \sqrt{5{\cdot}4}$. Well, then, I am sure you will admit that $BM = \sqrt{38{\cdot}4}$, and $PM = \sqrt{5{\cdot}4}$, when the diameter of the circle = 8. Now, $BM - PM = BP$. But how are we to compute the arithmetical value of B P, without "*wielding that indispensable instrument of science, Arithmetic*"? B M and P M being incommensurables, must we not first extract their roots? Can we directly subtract $\sqrt{5{\cdot}4}$ from $\sqrt{38{\cdot}4}$? I certainly don't know how this is to be accomplished. It may be said that $BP = FH$, therefore, $BP = \sqrt{15}$, but that is not the question. What is wanted is the arithmetical difference between the values of BM and PM = BP. Now, $\sqrt{38{\cdot}4} = 6{\cdot}196\ldots$; and $\sqrt{5{\cdot}4} = 2{\cdot}323\ldots$; and $6{\cdot}196 - 2{\cdot}323 = 3{\cdot}873$ nearly. But it may be demonstrated, that $TP = \frac{7}{24}(BT) = \frac{7 \times 3}{24} = {\cdot}875$; therefore, $(BF + TP) = (3 + {\cdot}875) = 3{\cdot}875 = BP$, and is greater than H F. How is this? Is Applied Mathematics at fault? Mathematics have been an important branch of your studies. Can you explain this enigma? Is it not a proof, that by not permitting that "*indispensable instrument of science, Arithmetic*" to assert its supremacy, we may be led into the most serious errors in our Mathematical calculations?

In my Letter of yesterday, with reference to the diagram enclosed therein (*see Diagram XIX.*), I said that $EN : GD :: ED : GO$. Now, when $AO = 4$, then, $OB = 3$, and, $AO : OB :: OB : OD$; that is, $4 : 3 :: 3 : 2{\cdot}25$; therefore, $OD = 2{\cdot}25$; and it follows, that $AO + OD = 4 + 2{\cdot}25 = 6{\cdot}25 = AD$. But, A D is bisected at E; therefore, $\frac{AD}{2} = \frac{6{\cdot}25}{2} = 3{\cdot}125 = ED$, and AOB and EDN are similar triangles. Now, $\frac{3}{4}(EB) = \frac{3 \times 3{\cdot}125}{4} = 2{\cdot}34375 = DN$; therefore, $(ED^2 + DN^2) = (3{\cdot}125^2 + 2{\cdot}34375^2) = (9{\cdot}765625 + 5{\cdot}4931640625) = 15{\cdot}2587890625 = EN^2$; therefore, $\sqrt{15{\cdot}2587890625} = 3{\cdot}90625 = EN$, and is equal to the half of A M; that is, = half the hypothenuse of the right-angled triangle A D M. But, when $AO = 4$, then, $OG = 1$, and $GD = 1{\cdot}25$; and it follows, that $EN : GD :: ED : OG$; that is, $3{\cdot}90625 : 1{\cdot}25 :: 3{\cdot}125 : 1$. Hence:

$\frac{3\cdot90625}{1\cdot25}$ and $\frac{3\cdot125}{1}$ are equivalent ratios, and both express the ratio of circumference to diameter, in every circle.

I called your attention to the fact, that in the right-angled triangle B O E, the sides B O and O E, which contain the right angle, are in the ratio of 24 to 7, by construction. Hence: B O is to B E as the perimeter of a regular hexagon to the circumference of its circumscribing circle.

You will observe that when A D = 6·25, a square on E N = 15·2587890625. From this fact we get an interesting problem. Let *a* denote any finite number, say 60, and represent the area of a circle; and let *b* denote the area of another circle and contain *a* 15·2587890625 times. Find the radii of the respective circles, and construct a geometrical figure furnishing the proof. The *only difficulty* in the solution of this problem is, that we must first find the true arithmetical value of π.

I remain, my dear Sir,
Very respectfully yours,
JAMES SMITH.

J. M. WILSON, ESQ.

JAMES SMITH *to* J. M. WILSON, ESQ.

BARKELEY HOUSE, SEAFORTH,
28th October, 1868.

DEAR SIR,

I *must* tell you courteously, but most distinctly, that my communication of the 26th inst., is *not*—as you say—"*based on an error:*" and I *may* tell you how it happened that you came to make this *assertion*. You failed to discover, what you might, and ought to have discovered, that the triangles O B P and K B M, in the diagram in "*Euclid at Fault*," (*See Diagram XIV.*), are not similar triangles, (*Is not a comparison of triangles an essential element in Geometry on your own shewing?*) and your *error* consists, in making an *assertion*, based upon the *assumption* that, they are similar triangles.

I may tell you that, as regards your own reputation, you will commit a great mistake, if you decline to read my communications; and as I write, *not* for information, no correspondence on your part is necessary.

Yours truly,
JAMES SMITH.

J. M. WILSON, ESQ.

BARKELEY HOUSE, SEAFORTH,
29*th October*, 1868.

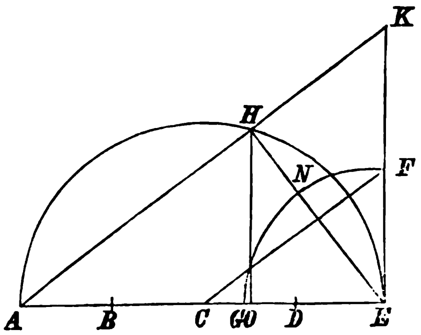
JAMES SMITH *to* J. M. WILSON, ESQ.

DEAR SIR,

As my object is not to seek, but to give information, I make no apology for again addressing you. I have no doubt you are engaged during leisure hours on your second part of "*Elementary Geometry*," which is to embrace the Geometry of the Circle and the applications of proportion to Plane Geometry. I should not be acting fairly with you, if I withheld the information I possess on these subjects, and so enable you to make the forthcoming work conformable with the knowledge of the age. This is my reason for again troubling you. If you *despise instruction*, and decline to read, the fault will be yours, not mine; and as you are a young man, the day may probably arrive when you will deeply regret such a course of conduct.

I construct this very simple geometrical figure in the following way.

From the point A, I draw a straight line of indefinite length, and mark off four equal parts A B, B C, C D, and D E; and with C as centre and C A or C E as interval, describe the semicircle



A H E. From the point E, I draw a straight line of indefinite length—somewhat longer than E C, but less than E B—at right angles to A E, and therefore tangental to the semicircle. I then bisect C D at the point G, and with E as centre and E G as interval, I describe the arc G N F; and join C F, producing the right-angled triangle F E C. It is self-evident, that the sides C E and E F which contain the right angle, are in the ratio of 4 to 3, by construction. From the point A, I draw a straight line parallel to C F, to meet E F, produced at the point K, and so construct the right-angled triangle K E A. It is self-evident, that K E A and F E C are similar right-angled triangles, by construction. From the point H, where A K the hypothenuse of the triangle K E A intersects the semicircle, I let fall the perpendicular H O, and join H E.

Now, conceive us to draw a straight line from the point G, perpendicular to A E, to meet the semicircle at a point, say X, and join X A; it is self-evident, that the triangle X G A, so constructed, would not be a similar triangle to the triangle H O A. But the triangles HOA, HOE, EHA, EHK, KEA, and FEC, although widely different in *direction*, are similar right-angled triangles; that is to say, in all of them the sides that contain the right angle are in the ratio of 3 to 4, by construction. Conceive the triangle F E C to be fixed. Then: It is self-evident, that no lines drawn from the point A, *within* the line A K to meet points in the line K E, would produce a similar triangle to the triangle F E C. In other words, conceive a line fixed at A, starting from the "*initial position*" A K and to revolve towards F, resting at various points between K and F: it is self-evident that no triangle so constructed would be similar to the triangle F E C. On the other hand, it is equally self-evident that, if a series of lines be drawn from the point A *without* the line A K, to meet E K produced, no triangles so constructed would be similar to the triangle F E C. Hence: It follows of necessity, that the point H in the semicircle, is the only point to which we can draw a straight line from the point A one extremity of the diameter of the semicircle, join it with the point E the other extremity of the diameter of the semicircle, and let fall the perpendicular H O, from H, the right angle, so as to divide the triangle K E A into three similar right-angled triangles, and all similar to the triangle F E C.

Now, let the length of the line A O be given, to find the length of the line O E. I have no hesitation in telling you, my dear Sir, that the formula necessary to demonstrate this theorem, has never occurred to you, any more than to Professor de Morgan. I gave to that gentleman, some years ago, a formula for finding the difference between the area of a square and the area of its circumscribing circle; but Professor de Morgan has chosen to take his stand in the ranks of that numerous class who despise "*wisdom and instruction.*" Well, then, the following is the formula for finding the length of O E, when the length of A O is given. $\left\{\left(AO + \frac{AO}{4}\right) + \frac{1}{4}\left(AO + \frac{AO}{4}\right)\right\} = AE$; and it is self-evident that, $AE - AO = OE$: and I may tell you that whatever finite value you may be pleased to put upon A O, if you take that value as representing the area of a square, the length of O E will represent the difference between the area of the square and the area of its circumscribing circle.

For example: Let squares be inscribed and circumscribed to a circle; and, by hypothesis, let the area of the inscribed square = 6. Then: The area of the circumscribing square = 12: and, $\{(6 + \frac{6}{4}) + \frac{1}{4}\left(6 + \frac{6}{4}\right) = 7{\cdot}5 + 1{\cdot}875 = 9{\cdot}375$ = area of the circle; and, the area of any circle is found by multiplying the area of its circumscribing square by $\frac{\pi}{4}$. But, $\left\{\left(6 + \frac{6}{4}\right) + \frac{1}{4}\left(6 + \frac{6}{4}\right)\right\} = \left\{6 \times 2 \times \frac{3{\cdot}125}{4}\right\}$, and this equation is exactly equal to the equivalent equation $\left\{\left(6 + \frac{6}{4}\right) + \frac{1}{4}\left(6 \times \frac{6}{4}\right)\right\} = \frac{6 \times 2}{1{\cdot}28}$. If you ask me how this happens, I have simply to say, that $\frac{3{\cdot}125}{4} = {\cdot}78125$; and, $\frac{{\cdot}78125}{1}$, and $\frac{1}{1{\cdot}28}$ are equivalent ratios; and it follows of necessity, that the product of any quantity multiplied by ·78125 is equal to the quotient of the same quantity divided by 1·28. If Professor de Morgan chooses to take *his* stand in the ranks of that numerous class, who despise "*wisdom and instruction,*" why, my dear Sir, should *you?* You are a young man. I am almost old enough to be your grandfather. There is, therefore, no presumption on my part, in giving you a word of well-intentioned advice. Pray take it before it is too late! You, with the talents with which Providence has gifted you, have a noble career of usefulness open to you, if you do but make a right use of those talents. Let me advise you! Be not a follower and imitator of such men as Professor de Morgan, and so, prostitute the gifts with which Providence has blessed you!

Let A O = 5. Then: $\left\{\left(AO + \frac{AO}{4}\right) + \frac{1}{4}\left(AO + \frac{AO}{4}\right)\right\} = \{(5 + 1{\cdot}25) + 1{\cdot}5625\} = 7{\cdot}8125 = AE$. $(AE - AO) = (7{\cdot}8125 - 5) = 2{\cdot}8125 = OE$. $(AO \times OE) = (5 \times 2{\cdot}8125) = 14{\cdot}0625 = OH^2$; therefore, $\sqrt{14{\cdot}0625} = 3{\cdot}75 = HO$. Then, by analogy or proportion, $AO : OH :: OH : OE$, that is, $5 : 3{\cdot}75 :: 3{\cdot}75 : 2{\cdot}8125$; and it follows of necessity, that 3·75 and 2·8125 are the true arithmetical values of H O and O E. Then: $(HO^2 + OE^2) = (3{\cdot}75^2 + 2{\cdot}8125^2) = (14{\cdot}0625 + 7{\cdot}91015625) = 21{\cdot}97265625 = HE^2$; therefore, $\sqrt{21{\cdot}97265625} = 4{\cdot}6875 = HE$. $(HO^2 + AO^2) = (3{\cdot}75^2 + 5^2) = (140625 + 25) = 39{\cdot}0625 = HA^2$; therefore, $\sqrt{39{\cdot}0625} = 6{\cdot}25 = HA$. By analogy or proportion, $HA : HE :: HE : HK$; that is, $6{\cdot}25 : 4{\cdot}6875 :: 4{\cdot}6875 : 3{\cdot}515625$; therefore, $HK = 3{\cdot}515625$. Hence: When $AO = 5$, then, $\left(\frac{HA}{2}\right)^2 = HA + HK$; that is, $\left(\frac{6{\cdot}25}{2}\right)^2 = (6{\cdot}25 + 3{\cdot}515625) = 9{\cdot}765625 = \pi^2$; therefore, $(KA^2 - AE^2) = (HE^2 + HK^2)$; that is, $(9{\cdot}765625^2 - 7{\cdot}8125^2) = (4{\cdot}6875^2 + 3{\cdot}515625^2)$, or, $(95{\cdot}367431640625 - 61{\cdot}03515625) = (21{\cdot}97265625 + 12{\cdot}359619140625) = 34{\cdot}332275390625 = KE^2$; therefore, $\sqrt{34{\cdot}332275390625} = 5{\cdot}859375 = KE$. Hence: $\frac{4}{5}(KA) = \frac{9{\cdot}7625625 \times 4}{5} = \frac{39{\cdot}0625}{5} = 7{\cdot}8125 = EA$; and, $\frac{3}{4}(EA) = \frac{7{\cdot}8125 \times 3}{4} = \frac{23{\cdot}4375}{4} = 5{\cdot}859375 = KE$; and it follows of necessity, that all the triangles in this simple but interesting geometrical figure, are similar triangles, and have the sides that contain the right angle in the ratio of 3 to 4.

Now, it is self-evident, that A O is a longer line than A G; and it is axiomatic, if not self-evident, that $AG = 5$, when A E equal 8. Well, then, by hypothesis, let $AO = 5{\cdot}12$. Then: $\left\{\left(AO + \frac{AO}{4}\right) + \frac{1}{4}\left(AO + \frac{AO}{4}\right)\right\} = \left\{\left(5{\cdot}12 + \frac{5{\cdot}12}{4}\right) + \frac{1}{4}\left(5{\cdot}12 + \frac{5{\cdot}12}{4}\right)\right\} = (6{\cdot}4 + 1{\cdot}6) = 8 = AE$: $(AE - AO) = (8 - 5{\cdot}12) = 2{\cdot}88 = OE$: and, $(AO \times OE) = (5{\cdot}12 \times 2{\cdot}88) = 14{\cdot}7456 = HO^2$; therefore, $\sqrt{14{\cdot}7456} = 3{\cdot}84 = HO$. Then, by analogy or proportion, A O :

H O : : H O : O E ; that is, $5{\cdot}12 : 3{\cdot}84 :: 3{\cdot}84 : 2{\cdot}88$; and it follows of necessity, that 3·84 and 2·88 are the true arithmetical values of H O and O E, when $A\,O = 5{\cdot}12$. Then : $(H\,O^2 + O\,E^2) = (3{\cdot}84^2 + 2{\cdot}88^2) = (14{\cdot}7456 + 8{\cdot}2944) = 23{\cdot}04 = H\,E^2$; therefore, $\sqrt{23{\cdot}04} = 4{\cdot}8 = H\,E$: and, $(H\,O^2 + A\,O^2) = (3{\cdot}84^2 + 5{\cdot}12^2) = (14{\cdot}7456 + 26{\cdot}2144) = 40{\cdot}96 = H\,A$; therefore, $\sqrt{40{\cdot}96} = 6{\cdot}4 = H\,A$. Then, by analogy or proportion, $H\,A : H\,E :: H\,E : H\,K$; that is, $6{\cdot}4 : 4{\cdot}8 :: 4{\cdot}8 : 3{\cdot}6$; and it follows, that $(H\,A + H\,K) = (6{\cdot}4 + 3{\cdot}6) = 10 = K\,A$. Hence: We get the following equation :—$(K\,A^2 - E\,A^2) = (H\,E^2 + H\,K^2)$; that is, $(10^2 - 8^2) = (4{\cdot}8^2 + 3{\cdot}6^2)$ or, $(100 - 64) = (23{\cdot}04 + 12{\cdot}96) = 36$, and is equal to $K\,E^2$.

Now, let the length of $A\,O = 5{\cdot}12$, and let the area of an inscribed square to a circle $= 5{\cdot}12$. Then $2\,(5{\cdot}12) = 10{\cdot}24 =$ area of a circumscribing square to the circle. $\frac{5{\cdot}12}{4} = 1{\cdot}28$, and, $\frac{10{\cdot}24}{1{\cdot}28} = 10{\cdot}24 \times \frac{3{\cdot}125}{4}$; and this equation $= (A\,O + O\,E) = 8$, and it follows of necessity, that when the area of a square $= 5{\cdot}12$, the area of its circumscribing circle $= 8$, which makes 8 circumferences = 25 diameters in every circle, and $\frac{25}{8} = 3{\cdot}125$, the true arithmetical value of π.

30th October.

I had written so far yesterday, but deferred its completion until the receipt of this morning's letters. I thought it just possible that my note of the 28th might have induced you to change the course indicated by your laconic note of the 27th, and that you might have intimated it. I warn you again that you are playing a dangerous game, as regards your own reputation.

I remain,
Yours respectfully,
JAMES SMITH.

J. M. WILSON, Esq.

JAMES SMITH *to* J. M. WILSON, ESQ.

BARKELEY HOUSE, SEAFORTH,
31st October, 1868.

DEAR SIR,

If my health permit, I intend—before the next meeting of "*The British Association for the Advancement of Science*"—to bring out a work which will probably be entitled "*The Geometry of the Circle, and the true ratio of diameter to circumference, demonstrated by angles*." I shall shew that all existing Mathematical Tables of the Sines and Cosines of angles, are fallacious. How happens it that the Astronomers of the day can neither agree among themselves, or with those of bygone times, as to the distance of the sun from the earth? The cause is plain enough! How is it possible they can discover the sun's distance, while they remain ignorant of the Geometry of the circle, and consequently, *at fault* as to the ratio of diameter to circumference?

It is my present intention to bring out this work in the form of a series of Letters addressed to *you*, as a living authority and writer on such subjects; and it is probable I may give an introductory chapter in the form of a Letter to the Mathematical President Elect of the British Association, Professor Stokes.

Your laconic note of the 27th inst., and subsequent silence with reference to my reply of the 28th, have led me to this decision.

You will please understand that I do not wish you to give yourself the slightest trouble about me, and whether you do, or do not, read my Letters, is to me a matter of perfect indifference. You might, however, think I was treating you unfairly, if I were not to acquaint you with my intention.

I may say, in conclusion, that I cannot help thinking the day will come, when you will regret having chosen to take your place in the ranks of those who despise instruction.

I remain, dear Sir,

Yours truly,

J. M. WILSON, ESQ. JAMES SMITH.

J. M. WILSON, ESQ. *to* JAMES SMITH.

RUGBY, *October* 31*st*, 1868.

DEAR SIR,

I cannot say I have read either of your last communications, but as I turned over the pages of your letter this morning, one expression struck me. You implore me not to despise instruction from one who is old enough to be my grandfather. I was in error on this point; I thought you were still a young man, and I therefore wrote laconically, not to say curtly, to you; and without that respect due from a young man to an old one. I beg to apologise for this.

At the same time, I must tell you that your Letter about the similar triangles was distinctly in error, at a point which I described to you in my first Letter to you. I have read your early Letters on the area of a circle, and in them also there are distinctly unfair assumptions which you have not noticed.

I do not despise instruction from any one. I reserve, however, the right of judging, by the rules of logic and common sense, the validity of any reasoning that is sent me: and yours will not stand the test.

I wish for your own happiness you could be brought to believe this.

Very truly yours,
J. M. WILSON.

I again request that this correspondence may cease.

JAMES SMITH *to* J. M. WILSON, ESQ.

BARKELEY HOUSE, SEAFORTH,
2nd November, 1868.

DEAR SIR,

I beg to acknowledge the receipt of your favour of Saturday, and can sincerely assure you that I never felt offended with you, or that any apology on your part was necessary for having written to me somewhat laconically; and I only referred to my seniority as an apology for offering you advice.

I "*reserve the right of judging by the rules of logic and common sense*," and admit every man's right to do so likewise. It would be better for science, if "*the rules of logic and common sense*" were more strictly adhered to; but I must tell *you*, "courteously, but most distinctly," that I dispute your conclusion, that my reasoning will not stand the test of "*logic and common sense.*" "*The two eyes of exact science*," are not—as Professor de Morgan says—"*Mathematics and logic*," but Geometry and logic; and it has long been the rule of Mathematicians to make Mathematics, not only over-ride Geometry, but even over-ride "*that indispensable instrument of science, Arithmetic*," the root and foundation of all Mathematics. There is a sense in which Geometry may be said to be a science, *per se*, and quite independent of Mathematics. A living "*recognised Mathematician*," J. R. Young, says in his preliminary remarks, on the "Principles of Algebra":—"*It would be advisable for a learner, after some acquaintance with a book or two of Euclid, to combine Geometry and Algebra together. The former subject, as it is independeut of all previous knowledge of science, may be entered upon by anyone who is familiar with ordinary language; he may be ignorant of even the multiplication table, and yet be able to master all the propositions in Euclid.*" I have shewn, in my Letter to you of the 27th October, that by applied Mathematics, we may apparently make the opposite sides of a parallelogram unequal. What can Algebra do towards shewing this? Well, then, Euclid is at fault, and could not be otherwise than at fault, without travelling out of the domain of pure Geometry; inasmuch as he does not shew that his fifth book is inapplicable, when mathematically applied to Geometry, in the case of *incommensurables*. How happens it that Mathematicians have never made this discovery?

Does not this explain the fallacy in the following quotation, which I take from the appendix to the fifth and sixth books of "*Euclid's Elements of Plane Geometry*," by W. D. Cooley, A.B.? The same proposition (Prop 13: Book 6) which enables us to find a mean proportional between two given lines, will also enable us to find a mean proportional between the first and second, and between the second and third; and thus to interpolate mean proportionals between the terms to any extent. But, to find two mean proportionals, or, A and B being given, to find X and Y, so that A : x : : y : B, is a problem beyond the reach of Plane Geometry. Now, I can shew you, my dear Sir, that from A and B given to find x and y, so that A : x : : y : B, is not a problem beyond the reach of Plane Geometry. Cooley would be right if π were arithmetically *indeterminate*, and *proportion* mathematically applicable to *incommensurables*, under all circumstances.

I remain, dear Sir,
Yours very respectfully,
J. M. WILSON, ESQ. JAMES SMITH.

BARKELEY HOUSE, SEAFORTH,
4th November, 1868.

DEAR SIR,

When I attended the last meeting of "*The British Association for the Advancement of Science*," I was introduced to a gentleman—a first-rate Mathematician—not much my junior, and presented him with a copy of "*Euclid at Fault*." The morning I left Norwich, he called at the Hotel where I was staying, to see a friend, and this gave me the opportunity of having about ten minutes' conversation with him. He mentioned some of the difficulties that occurred to him with reference to Euclid's fifth book on Proportion, and asked me if there was any speciality about the number 8, which led me to shortly give him a THEORY on the Geometry of commensurable right-angled triangles, which I discovered in 1860. It was quite new to him, and he was greatly astonished with it, and thanked me for bringing it under his notice. On parting, I promised to send him some of my pamphlets on my return home. I did so, and the following is a copy of his note acknowledging their receipt.

September 3rd, 1868.

MY DEAR SIR,

I thank you much for your pamphlets ; I have read a part of one—and so far your reasoning appears all right—but I will write you again. I cannot understand why any Mathematican should for one moment hesitate to investigate any problem which has been worked out by another, especially when it is a problem of considerable interest, because prejudice ought to have no place whatever in the Mathematical regions.

Again thanking you—I have just time to subscribe myself.

Yours truly,
M.——

QUEEN HOTEL, HARROGATE,
5th September, 1868.

MY DEAR SIR,

Your note of the 3rd inst. has been forwarded to me here, and came into my hands this morning.

I much regretted, that time did not permit me—before we were obliged to part at Norwich—to go a little deeper into the *theory* of commensurable right-angled triangles, in which you obviously felt much interested, and I am induced to jot down for your consideration, a few observations on this very interesting, and—to Science—very important subject.

Well, then, the following is a *general rule* for finding a commensurable right-angled triangle.

Let A and B denote any two finite numbers, either consecutive or non-consecutive. Then: If twice the product of A and B and the difference of their squares denote sides of a right-angled triangle and contain the right angle, the triangle is commensurable.

For example: Let $A = 7$, and $B = 9$. Then: $2\,(A \times B) = 2\,(7 \times 9) = (2 \times 63) = 126$: and, $(B^2 - A^2) = (9^2 - 7^2) = 81 - 49 = 32$. Now, if in a right-angled triangle, the sides that contain the right angle be 126 and 32. Then: $\sqrt{(126^2 + 32^2)} = \sqrt{(15876 + 1024)} = \sqrt{16900} = 130$ = hypothenuse, and the triangle is commensurable. Again: Let $A = 6$, and $B = 9$. Then: $2\,(A \times B) = 2\,(6 \times 9) = (2 \times 54) = 108$: and, $(B^2 - A^2) = (9^2 - 6^2) = 81 - 36 = 45$; therefore, $\sqrt{(108^2 + 45^2)} = \sqrt{(11664 + 2025)} = \sqrt{13689} = 117$ = hypothenuse, and it follows, that if 108 and 45 denote sides of a right-angled triangle and contain the right angle, the triangle is commensurable. In both examples, the hypothenuse is equal to the longer of the sides containing the right angle, plus the square of the difference between A and B. Hence: it follows of necessity, that if A and B be consecutive numbers, and be given to find a commensurable right-angled triangle, the hypothenuse of the triangle thus obtained, is equal to the longer of the sides containing the right angle, plus the square of $1 = 1$. Do not Mathematicians seem to lose sight of the fact, and its consequences, that 1, 1^2, and $\sqrt{1}$ are all arithmetical expressions equivalent to unity.

If the given numbers to find a commensurable right-angled triangle be—	The sides of the triangles will be—
1 and 2.	3, 4 and 5.
2 and 3.	5, 12 and 13.
3 and 4.	7, 24 and 25.
4 and 5.	9, 40 and 41.

and so on, *ad infinitum.*

Again: The two shorter sides of any of these triangles, plus the shortest side of the adjacent following triangle, is equal to the middle side of the latter triangle; that is to say, $3 + 4 + 5 = 12$: $5 + 12 + 7 = 24$: $7 + 24 + 9 = 40$: and so on, *ad infinitum.*

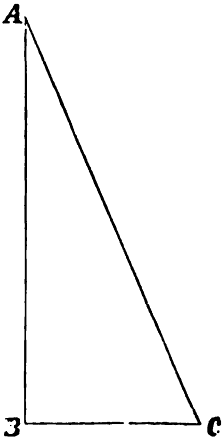
Now, with these facts before us, the following question arises:— Can a geometrical figure be constructed, in which there shall be isolated and exhibited, three right-angled triangles, of which the sides of one shall be 3, 4, and 5 : the sides of another 5, 12, and 13. and the sides of the third 7, 24, and 25, or in these proportions? I answer—Yes! But, not only so, this very figure can be constructed so as to contain—isolated and exhibited—a circle and a square of exactly the same superficial area, and demonstrable to be so, beyond the possibility of dispute or cavil by any candid Geometer and Mathematician. Passing this by, however, for the present, as it would carry me beyond the limits of a Letter, I will proceed to direct your attention to some truths which I think will surprise you, even more than the foregoing.

Let $A = 6$, and $B = 8$, and let 6 and 8 be given numbers to find a commensurable right-angled triangle. Then: $2(A \times B) = 2(6 \times 8) = (2 \times 48) = 96$: and, $(B^2 - A^2) = (8^2 - 6^2) = (64 - 36) = 28$. Now, if 96 and 28, which are in the ratio of 24 to 7, denote the sides that contain the right angle in a right-angled triangle, then, $(96^2 +$

28^2) = (9216 + 784) = 10000 = the square of the hypothenuse ; therefore, $\sqrt{10000}$ = 100 = hypothenuse, and is greater than the longer of the sides that contain the right angle by the square of the difference between A and B, when A = 6 and B = 8 : and we get by analogy or proportion, 96 : 28 : : 24 : 7 : and by alternation, ·96 : 24 : : 28 : 7. Again : Let A = 96, and B = 28, and let 96 and 28 be given numbers, to find a commensurable right-angled triangle. Then : 2 (A × B) = 2 (96 × 28) = 2 × 2688 = 5376 : and, (A^2 — B^2) = (96^2 — 28^2) = 9216 — 784 = 8432. Now, if 5376 and 8432 denote the sides that contain the right angle in a right-angled triangle, then, 5376^2 + 8432^2 = 28901376 + 71098624 = 100000000, the square of the hypothenuse ; therefore, $\sqrt{100000000}$ = 10000 = hypothenuse. But the square of the difference between A and B = $(96 - 28)^2$ = 68^2 = 4624. Hence : In the commensurable right-angled triangle thus obtained, the square of the difference between A and B, plus the shorter of the sides that contain the right angle = 4624 + 5376 = 10000 = hypothenuse. Now, 8 is a common multiple of 4624 and 5376, therefore, $\frac{4624}{8}$ = 578, and, $\frac{5376}{8}$ = 672 ; and we get the analogy or proportion, 4624 : 5376 : : 578 : 672 : and by alternation, 4624 : 578 : : 5376 : 672. I may be asked :—What inference do you draw from these facts ? The inference is this. They stand inseperably connected with our system of Logarithms to the base 10.

Let A B C denote a right-angled triangle of which the sides A B and B C which contain the right angle, are 24 and 7, by construction. Then : A B^2 + B C^2 = 24^2 + 7^2 = 576 + 49 = 625 = A C^2 ; therefore, $\sqrt{625}$ = 25 = A C the hypothenuse.

Now, $\frac{BC}{AC} = \frac{7}{25}$ = ·28 is the sine of the angle A, and cosine of the angle C : and $\frac{AB}{AC} = \frac{24}{25}$ = ·96 is the sine of the angle C, and cosine of the angle A. The Logarithm corresponding to the natural number ·28 is 9·4471580, and this is the Log-sin. of the angle A, and Log.-cos of the angle C. The



Logarithm corresponding to the natural number ·96 is 9·9822712, and this is the Log.-sin of the angle C, and Log.-cos. of the angle A.

Well, then, let the length of A C, the hypothenuse in the triangle A B C, be represented by any finite arithmetical quantity, say 666, and be given to find the length of the other two sides, and prove that those sides are in the ratio of 7 to 24.

Then :

As Sin. of angle B = Sin. 90°..........................Log. 10·0000000
: the given side A C = 666................................Log. 2·8234742
: : Sin. of angle A.............. ....Log. 9·4471580

12·2706322
10·0000000

: the required side B C
= 666 × ·28 = 186·48................................Log. 2·2706322

Again :

As Sin. of angle B = Sin. 90° ....Log. 10·0000000
: the given side A C = 666 ....Log. 2·8234742
: : Sin. of angle C ....Log. 9·9822712

12·8057454
10·0000000

: the required side A B
= 666 × ·96 = 639·36...............................Log. 2·8057454

Hence :

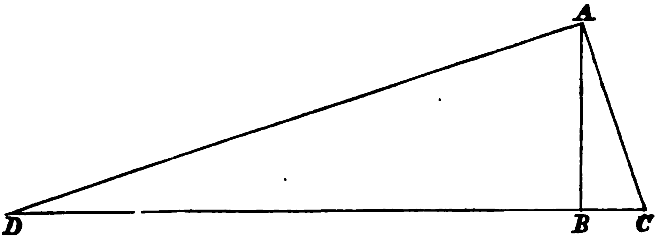
B C : A B : : 7 : 24 ; that is, 186·48 : 639·36 : : 7 : 24.
B C : A C : : 7 : 25 ; that is, 186·48 : 666 : : 7 : 25.
A B : A C : : 24 : 25 ; that is, 639·36 : 666 : : 24 : 25.

Therefore :

$·28^2 + ·96^2 = ·0784 + ·9216 = 1 =$ unity ; and harmonizes with the requirement of the trigonometrical axiom, $\text{Sin.}^2 + \text{Cos.}^2 =$ unity in every right-angled triangle. I think you will perceive that, if instead of the whole numbers 6 and 8, we make the fractions $\frac{6}{10}$ and $\frac{8}{10}$ = ·6 and ·8 our starting point, to find a commensurable right-angled triangle, these numbers give us ·28 and ·96 as the sides that contain the right angle, which are the sine and cosine of the acute angle in a triangle of which the sides that contain the right angle are in the ratio of 7 to 24. We may then take ·28 and ·96 as given numbers to find another commensurable right-angled triangle.

and if so, the hypothenuse of this triangle will be unity = 1. And so on we may proceed, *ad infinitum*, and the hypothenuse of every successive triangle will be a constant quantity = unity = 1.

In this geometrical figure, let A B C be a right-angled triangle, of which the sides B C and A B are 5 and 12, by construction. From the angle A, draw a straight line at right angles to A C, to meet C B produced at the point D. Then: The triangles on each side of A B are similar to the whole triangle D A C and to each other.



Proof: When B C = 5, and A B = 12, we get the analogy or proportion, B C : A B : : A B : B D, that is, 5 : 12 : : 12 : 28·8; therefore, (B D × B C) = A B², that is, (28·8 × 5) = 144 = A B². But, the triangle A B D is right angled at B; therefore, (A B² + B D²) = (12² + 28·8²) = (144 + 829·44) = 973·44 = A D²; therefore, $\sqrt{973{\cdot}44}$ = 31·2 = A D, and A B D is a commensurable right-angled triangle.

Now, let the length of A D, the hypothenuse in the right-angled triangle A B D, be represented by any finite arithmetical quantity, say 666, and be given to find the lengths of the sides A B and B D, and prove that they are in the ratio of 5 to 12.

Then: $\frac{AB}{AD} = \frac{12}{31{\cdot}2} = {\cdot}384615$ is the sine of the angle D, and cosine of the angle D A B. $\frac{BD}{AD} = \frac{28{\cdot}8}{31{\cdot}2} = {\cdot}923076$ is the sine of the angle D A B, and cosine of the angle D: and in both cases the sines and cosines are recurring decimals. The Logarithm corresponding to the natural number ·384615 is 9·5850263, and this is the Log.-sin. of the angle D, and Log.-cos. of the angle D A B. The Logarithm corresponding to the natural number ·923076 is 9·9652375, and this is the Log.-sin. of the angle D A B, and Log.-cos. of the angle D.

Hence:

As Sin. of angle B = Sin. 90°	Log.	10·0000000
: the given side A D = 666	Log.	2·8234742
: : Sin. of angle D	Log.	9·5850263
		12·4085005
		10·0000000
: the required side A B = 666 × ·384615 = 256·15359	Log.	2·4085005

Again:

As Sin. of angle B = Sin. 90°	Log.	10·0000000
: the given side A D = 666	Log.	2·8234742
: : Sin. of angle D A B	Log.	9·9652375
		12·7887117
		10·0000000
: the required side B D = 666 × ·923076 = 614·768616	Log.	2·7887117

Therefore:

A B : B D : : 5 : 12; that is, 256·15359 : 614·768616 : : 5 : 12; and by alternation, A B : 5 : : B D : 12; that is, 256·15359 : 5 : : 614·768616 : 12.

Hence: $(AD^2 - BD^2) = (AC^2 - BC^2)$, and this equation = AB^2; and it follows, that if in a right-angled triangle, a straight line be drawn from the right angle perpendicular to its opposite side, it is only when the triangles on each side of it are commensurable right-angled triangles, that the theorem of Euclid : Prop. 8, Book 6 holds good. Under any other circumstances it fails, when tested by "*that indisputable instrument of science, Arithmetic.*"

The fifth book of Euclid has ever been a puzzle to Mathematicians, and yet on reflection, the reason is plain enough. Euclid in attempting to make his theorems in this book of general and universal application, and therefore alike applicable to commensurables and incommensurables, has utterly failed. The 8th proposition is not the only one in the 6th book in which Euclid is at fault. The theorems of the 6th book rest for their proof on the 5th. You will observe that the 5th book is directly appealed to for the proofs of the first seven theorems in the 6th book; but I am warned that it is time to bring this Letter to a close.

In another communication I may point out, some of the inconsistencies into which Mathematicians are led by the teachings of Euclid. In the meantime,

Believe me, my dear Sir,
Very sincerely yours,
JAMES SMITH.

BARKELEY HOUSE, SEAFORTH.
12th September, 1868.

MY DEAR SIR,

I found my Letter of the 5th inst. running on to such a length, that I brought it to a close somewhat abruptly; and, indeed, I thought it better to defer proceeding further until my return home, and for the short period of my stay at Harrogate, take as much outdoor exercise as possible, and so reap all the advantage to be derived from the salubrious air of that delightful watering place.

Before proceeding to point out the inconsistencies into which Mathematicians are led, by the fallacious teachings of Euclid, I may direct your attention to certain facts, with which my last Letter might, and really ought to have concluded.

You will observe, that the arithmetical values of the sides of the commensurable right-angled triangle derived from the consecutive numbers 1 and 2, are 3, 4, and 5. This I call the *primary* commensurable right-angled triangle, since it is the smallest commensurable right-angled triangle, of which the sides can be arithmetically expressed in digits or whole numbers. With reference to this particular triangle, the *general rule* holds good; that is to say, the sum of 1 and 2, and twice the product of 1 and 2, give the arithmetical values of the sides that contain the right angle. But, there is one *peculiarity* about this particular triangle which holds good of no other, and to this peculiarity I must now direct your especial attention.

In the geometrical figure represented by the Diagram in my Letter to Dr. Hooker *(see Diagram XIV.)*, the sides O B and B T, in the right-angled triangle O B T, are in the ratio of 4 to 3, by construction; and when K B the diameter of the circle = 8, then, O B = 4; B T = 3; and O T = 5. Hence: The sum of O B and B T = (4 + 3) = 7. But, twice the product of O B and B T = twice the sum of O B, B T and O T; that is, $2(4 \times 3) = 2(4 + 3 + 5)$; and this equation $= 24$; and 7 and 24 are the arithmetical values of the sides that contain the right angle in the commensurable right-angled triangle, derived from the consecutive numbers 3 and 4.

Now, my dear Sir, no other right-angled triangle exists, or can exist, but that in which the sides that contain the right angle are represented by the digits 3 and 4, from which we can get the equation,—"*Twice the product of the sides that contain the right angle = twice the perimeter of the triangle.*" From this fact, it follows of necessity, that $(\mathrm{O\,B}^2 + \mathrm{B\,T}^2 + \mathrm{O\,T}^2) = 3\frac{1}{8}(\mathrm{O\,B}^2)$, that is, $(4^2 + 3^2 + 5^2) = 3\frac{1}{8}(4^2) = 50$: and on the THEORY that 8 circumferences of a circle $=$ 25 diameters, this equation $=$ area of the circle. But further: when the sides that contain the right angle in a right-angled triangle are 7 and 24, then $(7^2 + 24^2) = (49 + 576) = 625$ $=$ the square of the hypothenuse, and we get the following equation: $\left(\frac{\mathrm{O\,B}^2 + \mathrm{B\,T}^2 + \mathrm{O\,T}^2}{2}\right)^2 = 7^2 + 24^2 = 625$, when $\mathrm{O\,B} = 4$, and $\mathrm{B\,T} = 3$. Hence: when the sides that contain the right angle, in a right-angled triangle, are 3 and ·875, which are in the ratio of 24 to 7, then, $3^2 + {\cdot}875^2 = 9 + {\cdot}765625 = 9{\cdot}765625$; therefore, $\sqrt{9{\cdot}765625} = 3{\cdot}125 = 3\frac{1}{8}$, and this is the true value of π, and makes $\frac{3}{3{\cdot}125}$ the true expression of the ratio between the perimeter of every regular hexagon, and the circumference of its circumscribing circle.

Let OB, the radius of the circle, in the Diagram in "*Euclid at Fault,*" (*See Diagram XIV.*), be represented by the arithmetical expression $\sqrt{60}$. Then: $\frac{3}{4}(\mathrm{OB}) = \frac{3}{4}(\sqrt{60}) = \sqrt{\left(\frac{3^2}{4^2} \times 60\right)} = \sqrt{33{\cdot}75} =$ B T; and, $\frac{5}{4}(\mathrm{O\,B}) = \frac{5}{4}(\sqrt{60}) = \sqrt{\left(\frac{5^2}{4^2} \times 60\right)} = \sqrt{93{\cdot}75} =$ O T; and it follows of necessity, that $\mathrm{O\,B}^2 + \mathrm{B\,T}^2 + \mathrm{O\,T}^2) = 3\frac{1}{8}(\mathrm{O\,B}^2)$: that is, $(60 + 33{\cdot}75 + 93{\cdot}75) = 3\frac{1}{8}(60)$, and this equation $= 187{\cdot}5$, and on the THEORY that 8 circumferences of a circle $=$ 25 diameters $=$ area of the circle. But, on these values of the sides of the triangle O B T, we cannot get the equation:—"*Twice the product of* O B *and* B T $=$ *twice the sum of* O B, B T, *and* O T."

Now, by computation, twice the product of O B and B T $= 90$, when O B $\sqrt{60}$: twice the product of BT and OT $=$ 112·5 : twice the product of O B and O T $= 150$: the sum of the squares of O B, B T, and O T, $=$ 187·5 : and $100 (3\frac{1}{8}) =$ 312·5. Well, then, let the results of these computations be denoted by the symbols a, b, c, d, and e: that is, let $a = 90$: $b =$ 112·5 : $c = 150$: $d =$ 187·5: and $e =$ 312·5. Then: $\frac{5}{4}(a) = b$: $\frac{4}{3}(b) = c$: $2\frac{1}{12}(a) = d$: and d is a mean proportional between b and e. Hence: the equation $2\frac{1}{12}(90) = \sqrt{(112{\cdot}5 \times 312{\cdot}5)}$, and it follows, that in the analogy or proportion $b : d :: d : e$, d is the mean proportional between b and c. But, $a =$ the quadrant of a circle of circumference 360 : and, $25 . \left(\frac{a}{10}\right) = 72 \left(\frac{e}{100}\right)$; or, $25 \left(\frac{90}{10}\right) = 72 \left(\frac{312{\cdot}5}{100}\right)$; that is, $25 \times 9 = 72 \times 3{\cdot}125$, and this equation $= 225 = 2(b)$, and is equal to the circumference of a circle of which the diameter is 72 : and it follows, that 8 times 225 $=$ 25 times 72 $=$ 1800, and establishes the truth of the *theory*, that 8 circumferences $=$ 25 diameters in every circle, and makes $\frac{25}{8} = 3{\cdot}125$ the true arithmetical value of π.

The discoveries contained in the last paragraph—which are very remarkable—are of recent date, but I may tell you that at an early period of my enquiry into the ratio of diameter to circumference in a circle, I was led to adopt as a THEORY, that 8 circumferences $=$ 25 diameters in every circle, which makes $\frac{25}{8} = 3{\cdot}125$ the value of π, or in other words, the circumference of a circle of diameter unity $= 1$. I found the truth of the *theory* demonstrable by inductive and deductive reasoning, by constructive Geometry, and *finally* by logarithms. In another communication I may go into proofs at some length, and point out a few of the absurdities into which Mathematicians are led by the teachings of Euclid. In the meantime,

Believe me, my dear Sir,
Very sincerely yours,
JAMES SMITH.

To these communications I received the following reply :—

September 15*th*, 1868.

My Dear Sir,

I have just returned from Yorkshire, and found your very interesting Letter, for which I thank you; also for the one received this morning: when I have a little leisure I will write you again. The subjects are full of interest, and as far as I have seen, your reasoning is perfectly satisfactory, and certainly to me it seems to open up an interesting and new region in this department of numbers. So does Mr. Byrne's Dual Arithmetic. In short, I think Mathematicians ought to congratulate you on your researches—instead of not giving the subjects their proper attention. With kind regards,

Believe me,
yours truly,
M——.

I am not going to trouble *you* with any more of my Letters to this gentleman, but the following is a copy of his acknowledgment of the receipt of two of them.

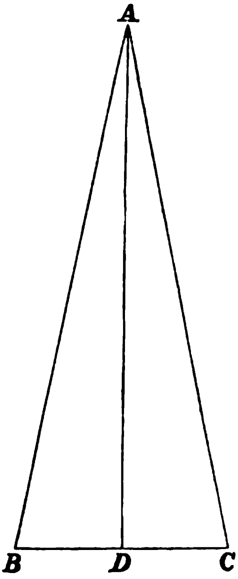
September 29*th*, 1868.

My Dear Sir,

Accept my thanks for your communications and "*The British Association in Jeopardy*" received this morning. I feel from what I have read that you are right—and at present I can't understand why Mathematicians have not thoroughly examined your reasonings long before this. The demonstration in pages 14 and 15 ought to have sufficed. I have not been very well, or I should have answered you before: this must be my apology. With kind regards,

Believe me,
yours truly,
M——.

The following is the demonstration referred to, which I gave in a Letter to the Editor of the *Athenæum*, in 1865, under the signature "*Nauticus*.'



"Let A B C represent one of 25 equal isosceles triangles inscribed in a circle. Then: $\frac{360}{25} = 14{\cdot}4 = 14^\circ\ 24'$ is the value of the angle A, contained by the two sides A B and A C; and the circular measure of this angle is $\frac{14^\circ\ 24' \times \pi}{180}$, which, on Mr. Smith's theory, is $\frac{864^\circ \times 3{\cdot}125}{180 \times 60} = \frac{2700}{10800} = {\cdot}25$. The circular measure of an angle of 90° is $\frac{90 \times \pi}{180} = \frac{3{\cdot}125}{2} = 1{\cdot}5625$, and is equal to 100 times $\left(\frac{{\cdot}25}{2}\right)^2 = 100\,({\cdot}125^2)$. But, 1·5625 is the value of the quadrant of a circle of radius 1, or, the semi-circumference of a circle of diameter unity. Therefore, $\frac{1{\cdot}5625}{{\cdot}25} = 6{\cdot}25$ = circumference of a circle of radius 1, and is equal to $2\,\pi$."

"I now beg to call your especial attention to the following facts:—On the Orthodox theory, $\pi = 3{\cdot}1416$ approximately; and, on this hypothesis, the circular measure of the angle A = $\frac{864^\circ \times 3{\cdot}1416}{10800} = \frac{2714{\cdot}3424}{10800} = {\cdot}251328$. The circular measure of an angle of $90^\circ = \frac{90 \times 3{\cdot}1416}{180} = \frac{3{\cdot}1416}{2} = 1{\cdot}5708$, and is equal to the quadrant of a circle of radius 1, or the semi-circumference of a circle of diameter unity, on the hypothesis that $\pi = 3{\cdot}1416$. But, $\frac{1{\cdot}5708}{{\cdot}251328} = 6{\cdot}25$, and is not equal to the Orthodox value of $2\,\pi$, but equal to the circumference of a circle of radius 1, on Mr. Smith's theory, and establishes beyond the possibility of doubt, by any reflective Mathematician, that $\frac{25}{8} = 3{\cdot}125$ (which is Mr. Smith's value of π) is the true arithmetical value of the circumference of a circle of which the diameter is unity."

But further: The number of degrees contained in an angle at the centre of a circle subtended by an arc equal to radius is $\frac{180^\circ}{\pi}$, and on the *theory* that 8 circumferences are exactly equal to 25 diameters $= \frac{180}{3 \cdot 125} = 57 \cdot 6 = 57^\circ\ 36'$, and the circular measure of an angle of $57^\circ\ 36'$ is $\frac{57^\circ\ 36' \times \pi}{180} = \frac{3456^\circ \times 3 \cdot 125}{180 \times 60} = \frac{10800}{10800} = 1 =$ radius.

I may tell you for your information—for I have no wish to conceal anything—that "Zadkiel" the Astrologer and Astronomer, gave this as an unanswerable demonstration in his almanac for the year 1866: and I have a letter of his in my possession, written as far back as 1860, in which he congratulates me on having discovered the true value of π. Oddly enough, however, he has since repudiated these ideas, and with them all his previous notions of Astronomy; and has recently published a work, which he entitles "*The New Principia*," in which he professes to prove that the earth is the stationary centre of the solar system, and the sun only 365·006 miles distant from the earth. When will wonders cease?

The *Athenæum*—or I should rather say, Professor de Morgan, writing as one of the editorial *we* of the leading scientific journal—has said of me:—"*We hope to have many a bit of sport with him in the future, as we have had in the past.*" He—the learned Professor—has had "*a bit of sport*" with you in the past, and if you decline to take "*a word of well-intended advice*," and neglect to make yourself master of "*the Geometry of the Circle and the application of proportion to Plane Geometry*," before you again appear in print, you may afford that clever, but dogmatic and unscrupulous critic, the opportunity of having "*a bit of sport*" with you in the future.

I remain, dear Sir,
Yours truly,
JAMES SMITH.

J. M. WILSON, ESQ.

BARKELEY HOUSE, SEAFORTH,
9th November, 1868.

DEAR SIR,

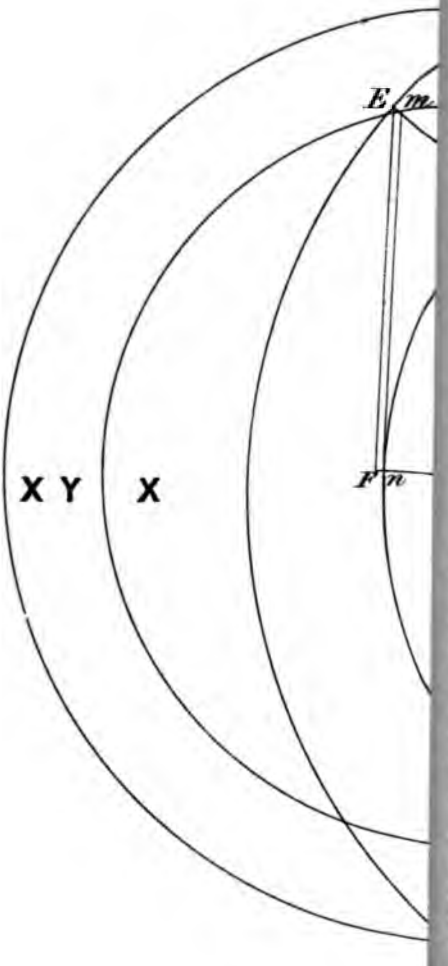
In my Letter to you of the 4th inst., I have given, what ought to be a convincing proof to the Mathematicians of the day, and will be to those of another generation, that existing Logarithmic Tables of Sines, Cosines, Log-sines, and Log-cosines, are fallacious; and before I conclude this communication, I shall furnish another proof.

If the sides of a right-angled triangle be 3, 4, and 5, or in these proportions: or in other words, in what I call the primary commensurable right-angled triangle, the obtuse angle is an angle of 53° 8′, and the acute angle an angle of 36° 52′. I shall make use of Hutton's Tables to prove these facts; and yet, in doing so, will demonstrate by "*the rules of logic and common sense*," that these Tables are fallacious. A gentleman with whom I was long in correspondence, assuming it to be impossible that Tables "*which have been calculated by experts in every country in Europe*" could be *at fault*, would have it that the former is an angle of $53° \, 8' - x$, and the latter an angle of $36° \, 52' + y$. This gentleman did not play the part of a fair and candid controversialist with me, and at one time I thought of making him the mathematical scape-goat in my next publication. I shall not do so now, and you may take to yourself the credit of having led me to change my mind, and confer that honour upon you.

PROBLEM.

From a right-angled triangle, of which the sides are 3, 4, and 5, or in these proportions, construct a diagram representing a geometrical figure, which shall contain—isolated and exhibited—two dissimilar and unequal right-angled triangles, so that the hypothenuse in both these triangles *may* be represented by the arithmetical expression $\sqrt{41}$.

The enclosed diagram *(see Diagram XX.)* solves the problem. It is obvious from mere inspection that, D B C and E F C must be the required triangles; since there are no other two triangles in the figure, of which the hypothenuses are radii of the same circle.


E
m
F
n
X
Y
X

CONSTRUCTION OF THE DIAGRAM.

Draw two straight lines of indefinite length at right angles, making B the right angle. From the point B mark off three equal parts, together equal to A B ; and from B mark off four of such equal parts, together equal to B C, and join A C. It is obvious that ABC must be a right-angled triangle, of which the sides that contain the right angle are in the ratio of 3 to 4. With B as centre and B C as interval, describe the circle X. With C as centre and C B as interval, describe the circle Y. With C as centre and C A as interval, describe the circle Z. With B as centre and the same interval, describe the circle X Y. Produce B A to meet and terminate in the circumference of the circle X Y at the point D. With C as centre and C D as interval, describe the circle X Z, and join D C and O C. The circles X and X Z intersect each other at the point E. From the point E draw a straight line, parallel to DB, to meet CB produced at the point F, and join EA, and so construct the right-angled triangle EFC. The circumference of the circle Z is intersected by the line F C at the point *n*. From *n* draw a straight line perpendicular to F C, and therefore tangental to the circle Z, to meet E C, the hypothenuse of the right-angled triangle E F C, at the point *m*, and so construct the right-angled triangle *m n* C.

By hypothesis, let $AB = 3$. Then : By construction, $BC = \frac{4}{3}(AB)$; therefore, $BC = 4$; and, $(AB^2 + BC^2) = (3^2 + 4^2) = (9 + 16) = 25 = AC^2$; therefore, $\sqrt{25} = 5 = AC$. But, $DB = AC$, by construction, and D B C is a right-angled triangle ; therefore, $(DB^2 + BC^2) = (5^2 + 4^2) = (25 + 16) = 41 = DC^2$; therefore, $DC = \sqrt{41}$. But, $EC = DC$, for they are radii of the circle X Z ; therefore, $EC = \sqrt{41}$. But, $Cn = CA$, for they are radii of the circle Z ; therefore, $Cn = 5$. But, *m n* C and A B C are similar right-angled triangles ; therefore, $\frac{3}{4}(Cn) = \frac{3}{4}(5) = \frac{3 \times 5}{4} = 3{\cdot}75 = mn$; and, $(mn^2 + nC^2) = (3{\cdot}75^2 + 5^2) = (14{\cdot}0625 + 25) = 39{\cdot}0625 = mC^2$; therefore, $\sqrt{39{\cdot}0625} = 6{\cdot}25 = mC$. But, A B C, *m n* C, and E F C, are similar right-angled triangles, and E C the hypothenuse of the triangle $EFC = \sqrt{41} = 6{\cdot}403124$; therefore, $\frac{4}{5}(EC) = \frac{4}{5}(6{\cdot}403124) = \frac{4 \times 6{\cdot}403124}{5} = \frac{25{\cdot}612496}{5} = 5{\cdot}1224992 =$

F C; and $\frac{3}{4}$ (F C) = $\frac{3}{4}$ (5·1224992) = $\frac{3 \times 5{\cdot}1224992}{4} = \frac{15{\cdot}3674976}{4}$ = 3·8418744 = E F; and, (E F^2 + F C^2) = (3·8418744^2 + 5·1224992^2) = (14·75999890537536 + 26·23999805400064) = 40·999996959376 = E C^2, and is a very close approximation to 41, the known and indisputable value of the area of a square on E C. But, O B = B C, for they are radii of the circle X; therefore, O B = 4; and O B C is a right-angled isosceles triangle; therefore, (O B^2 + B C^2) = ($4^2 + 4^2$) = (16 + 16) = 32 = O C^2; therefore, O C = $\sqrt{32}$, and is equal to a side of an inscribed square to the circle X; and $\sqrt{32}$ expressed *decimally* = 5·656854 approximately.

Now, O*p* C B is a quadrant of the circle X; therefore, B is an angle of 90°. But, the angles at the base of an isoceles triangle are equal; therefore the angle B O C = the angle O C B, and $\frac{\mathrm{O\,B}}{\mathrm{O\,C}}$ or $\frac{\mathrm{B\,C}}{\mathrm{O\,C}} = \frac{4}{5{\cdot}656854}$ = ·707106812, and ·707106812 is the sine of these angles. The Logarithm corresponding to the natural number ·707106812 is 9·8494950, and this is the Log.-sin. of the angles B O C and O C B; and B O C and O C B are angles of 45°.

Let the length of O C the hypothenuse of the right-angled isosceles triangle O B C, be represented by any finite arithmetical quantity, say 777, and be given to find the length of O B or B C, and prove that the ratio of O B to O C, or, B C to O C, is as 4 to 5·656854.

Then:

As Sin. of angle B = Sin. 90° .... Log. 10·0000000
: the given side O C = 777 .... Log. 2·8904210
: : Sin. of angle B O C or O C B = Sin. 45°......... Log. 9·8494950
12·7399160
10·0000000
: the sides O B and B C =
·707106812 × 777 = 549·421992924 .... Log. 2·7399160

Hence:

O B or B C : O C : : 4 : 5·656854, that is, 4 : 5·656854 : : 549·421992924 : 776 999999590325274; and it follows of necessity, that $\sqrt{\cdot 5}$ = ·707106812 is the arithmetical value of the Sine of an angle of 45°. In Hutton's Tables, the Sine and Cosine of an angle of 45° is given as ·7071068, and is equal to $\sqrt{\cdot 5}$ to the seventh place of decimals.

You do not read my communications, but this does not prevent me from supposing you to put the following question :—Where is the Mathematician who ever disputed or doubted that ·7071068 is the true value of the Sine of an angle of 45°? I may put a counter question :—Where is the Mathematician who ever gave the proof of it that I have now given? I may tell you that it never entered into your Mathematical philosophy to discover this proof. You will discover, and probably even in my time, the folly of "*having chosen to take your place in the ranks of that numerous clase who despise wisdom and instruction.*" If you had only as much regard for your own Mathematical reputation, as you profess you have for my happiness, I cannot help thinking it would not only be better for the interests of science, but in the long run add materially to your own happiness.

I shall now proceed to prove, that in the similar right-angled triangles E F C, $m n$ C, and A B C, the acute angle C which is common to the three triangles, is an angle of 36° 52′, and the obtuse angles, angles of 53° 8′, and, in doing so, will make use of Hutton to prove Hutton *at fault.*

Let A B = 3, and B C = 4. Then: A C = 5 : D B = 5 : $(DB^2 + BC^2) = 41$; therefore, D C = $\sqrt{41}$. Now, E C = D C, for they are radii of the same circle. But, E F is obviously a shorter line than O B ; and O B = B C, for they are radii of the same circle ; and F and B are right angles. Hence : E F C and D B C are right-angled triangles, and have their sides that subtend the right angle equal ; but they are not similar and equal triangles. Now, $\frac{AB}{AC} = \frac{3}{5} = {\cdot}6$, is the sine of the angle C ; and, $\frac{BC}{AC} = \frac{4}{5} = {\cdot}8$, is the sine of the angle A, in the triangle A B C. But, C n = C A = 5, for they are radii of the same circle, and $m n$ C and A B C are similar right-angled triangles ; therefore, $\frac{3}{4}(n\,C) = \frac{3}{4}(5) = 3{\cdot}75 = m n$: and, $\frac{5}{4}(n\,C) = 6{\cdot}25 = m\,C$; therefore, $\frac{m n}{m C} = \frac{3{\cdot}75}{6{\cdot}25} = {\cdot}6$, is the sine of the angle C ; and, $\frac{n C}{m C} = \frac{5}{6{\cdot}25} = {\cdot}8$, is the sine of the angle m, in the triangle $m n$ C : and it follows, that the sine and cosine of the angles m and C, in the triangle $m n$ C, are the same as the sine and cosine of the angles A and C in the triangle A B C.

Now, E C = D C = $\sqrt{41}$ = 6·403124. Well, then, let the obtuse angle E in the right-angled triangle E F C, by hypothesis, be an angle of 53° 8′, and the side E C subtending the right angle 6·403124 miles in length, and be given to find the lengths of the sides E F and F C, which contain the right angle. Then: 90 — 53° 8′ = 36° 52′ = the angle C.

Then: By Hutton's Tables—

As Sin. of angle F = Sin. 90° ....Log. 10·0000000
: the given side E C = 6·403124 miles ....Log. 0·8063919
: : Sin. of angle C = Sin. 36° 52′ ....Log. 9·7781186
10·5845105
10·0000000
: the required side E F
= 3·841586 miles..Log. 0·5845105

Again:
As Sin. of angle F = Sin. 90° ....Log. 10·0000000
: the given side E C = 6·403124 miles ....Log. 0·8063919
: : Sin. of angle E = Sin. 53° 8′...........................Log. 9·9031084
10·7095003
10·0000000
: the required side F C
= 5·122716 miles..Log. 0·7095003

On this shewing, the side E F is not to the side F C in the ratio of 3 to 4; neither is the side F C to the side E C in the ratio of 4 to 5. But, $\frac{E\,F}{E\,C} = \frac{3\cdot841586}{6\cdot403124} = \cdot5999549$ is the sine of the angle C, and the cosine of the angle E. $\frac{F\,C}{E\,C} = \frac{5\cdot122716}{6\cdot403124} =$ 8·000338 is the sine of the angle E, and cosine of the angle C. Now, ·5999549 and ·8000338 are the natural sines and cosines of angles of 36° 52′ and 53° 8′, as given in Hutton's Tables, and I think you will not dispute that I have proved by Hutton, that C and A in the triangle A B C are angles of 36° 52′ and 53° 8′.

Well, then, the triangles A B C, *m n* C, and E F C, are similar right-angled triangles, and the angle C is common to the three triangles; and I have proved that the sines and cosines of the angle C, in the two former triangles, are ·6 and ·8. But, according to Hutton, the natural sine of the angle C, in the triangle E F C, is *less* than ·6, and the natural cosine *greater* than ·8. How is this? Can the natural sines and cosines be different in similar right-angled triangles? Can trigonometry and practical or constructive Geometry be inconsistent with each other? Can a right-angled triangle have properties at variance with Logarithms? Your answer to all these questions must be :—Certainly not! What, then, is the explanation of this *apparent* inconsistency between Trigonometry and Geometry? Simply this! — 3·841586 and 5·122716 are *not* the true values of the sides E F and F C, in the triangle E F C, when E C the hypothenuse = $\sqrt{41}$, and this may be proved by Logarithms.

Let the length of the side E C in the triangle E F C = $\sqrt{41}$ = 6·403124, and be given to find the lengths of the other two sides. The angle C is an angle of 36° 52′, and the angle E an angle of 53° 8′. The sines of the angles C and E are ·6 and ·8, and I have proved that the Log.-sines corresponding to the natural numbers ·6 and ·8, are 9·7781513 and 9·9030900.

Then :

As Sin. of angle F = Sin. 90°............................Log. 10·0000000
: the given side E C = 6·403124 ....Log. 0·8063919
: : Sin of angle C = Sin. 36° 52′..........................Log. 9·7781513

10·5845432
10·0000000

: the side E F
= 6·403124 × ·6 = 3·8418744 ....Log. 0·5845432

Again :

As Sin. of angle F = Sin. 90° ....Log. 10·0000000
: the given side E C = 6·403124 ....Log. 0·8063919
: : Sin. of angle E = Sin· 53° 8′ ....Log. 9·9030900

10·7094819
10·0000000

: the side F C
= ·6403124 × ·8 = 5·1224992 ....Log. 0·7094819

Therefore:

E F : F C : : 3 : 4 ; that is, 3·8418744 : 5·1224992 : : 3 : 4.
F C : E C : : 4 : 5 ; that is, 5·1224992 : 6·403124 : : 4 : 5.
E F : E C : : 3 : 5 ; that is, 3·8418744 : 6·403124 : : 3 : 5.

Hence:

$$\frac{EF}{EC} = \frac{3{\cdot}8418744}{6{\cdot}403124} = {\cdot}6 = \text{sine of angle C.}$$

$$\text{and, } \frac{FC}{EC} = \frac{5{\cdot}1224992}{6{\cdot}403124} = {\cdot}8 = \text{sine of angle E.}$$

You cannot fail to perceive that, we arrive at the same values of E F and F C, the sides containing the right angle, in the triangle E F C, *by Logarithms*, as we arrive at in the demonstration given on page 4 of this communication, by common arithmetic. (*See page* 357.)

Again: Let the length of the side D C, which subtends the right angle in the right-angled triangle D B C be 60 miles, and be given to find the lengths of the sides B C and B D which contain the right angle, and prove that they are in the ratio of 4 to 5, the known ratio, by construction. The angle D is an angle of 38° 40′, therefore, 90° — 38° 40′ = 51° 20′ = the angle C.

Then, by Hutton's Tables:

As Sin. of angle B = Sin· 90° ....Log.	10·0000000
: the given side D C = 60 miles ....Log.	1·7781513
: : Sin of angle D = Sin. 38° 40′..........................Log.	9·7657330
	10·5738843
	10·0000000
: the side D B = 60 × ·6247885 = 37·48731 miles ....Log.	1·5738843

Again:

As Sin. of Angle B = Sin. 90° ....Log.	10·0000000
: the given side D C = 60 miles ....Log.	1·7781513
: : Sin. of angle C = Sin. 51° 20′ ....Log.	9·8925365
	11·6706878
	10·0000000
: the side D B = 66 × ·7807940 = 46·84764 miles ....Log.	1·6706878

But, 37·48731 miles, is not to 46·84764 miles, in the ratio of 4 to 5; and again, Hutton is *at fault.*

Now, we *know* that B C and D B are in the ratio of 4 to 5, by construction; and we *know* that when B C = 4 and D B = 5, that, D C = $\sqrt{41}$ = 6·403124. Then: $\frac{BC}{DC} = \frac{4}{6{\cdot}403124} =$ ·6246950, is the sine of the angle D; and $\frac{DB}{DC} = \frac{5}{6{\cdot}403124} =$ ·7808688, is the sine of the angle C. The Logarithm corresponding to the natural number ·6246950 is 9·7956649, and this is the Log.-sin of the angle D. The Logarithm corresponding to the natural number ·7808688 is 9·8923781, and this is the Log.-sin. of the angle C.

Then:

As Sin. of angle B = Sin. 90°	Log.	10·0000000
: the given side D C = 60 miles...	Log.	1·7781513
: : Sin. of angle D = Sin. 38° 40′	Log.	9·7956649
		11·5738162
		10·0000000
: the side B C = 60 × ·6246950 = 37·4817 miles	Log.	1·5738162

Again:

As Sin. of angle B = Sin. 90°	Log.	10·0000000
: the given side D C = 60 miles	Log.	1·7781513
: : Sin. of angle D = Sin. 51° 20′	Log.	9·8925781
		11·6707294
		10·0000000
: the side D B = 60 × ·7808688 = 46·852128 miles	Log.	1·6707224

Hence:

B C : D B : : 4 : 5; that is, 4 : 5 : : 37·4817 : 46·852125, arithmetically correct to the fifth place of decimals, and sufficiently accurate for all practical purposes. We can only get exact results, when dealing with commensurable right-angled triangles, and not even then, in all cases.

On the 10th August last, I wrote a long Letter to the gentleman referred to on the first page of this communication, and the following is a copy of his reply, which speaks for itself:—

15th August, 1868.

MY DEAR SIR,

I fear we are running into the old ruts over again. When I saw you light upon the angle 36° 52′, I foreboded what was coming.

The sine of 36° 52′ is ·5999549, and its Logarithm 9·7781186 to seven decimals. Now, *your* angle has for its sine $\frac{3}{5}$ = ·6, so that your angle is not 36° 52′, though very near it. I remember saying all this in our former correspondence, and do not wish to repeat it at any length, so that I totally disallow your calculations at page 3 of 10th August, when you say, Log.-sin of 36° 52′ = 9·77781513 ; the angle whose sine is ·6 is rather greater than 36° 52′.

Logarithms are only approximate values. This is evident by their construction ; by the calculation that determines them. If you look at the preface to Hutton—or any book on the subject—you will see that Logarithms are found only by an infinite series, unless they happen to be simple integers: thus, Log. 10 = 1, Log. 100 = 2 ; but the Log. of any intermediate number is inexpressible *exactly* in finite terms.

I am sorry to see you take so much trouble on my behalf, when I see "*ab origine*" that your labours must be fruitless. If Log.-sin. of ϕ be given (say 9·7781186), this gives ϕ only approximately, and can do no more. And what is more wonderful, you employ Logarithms which you declare to be erroneous, and which I allow to be inexact, to get out exact values ! So that in your Letter of 10th August, consisting of 18 sheets, there is a fatal flaw at the third, which brings me to a dead stand.

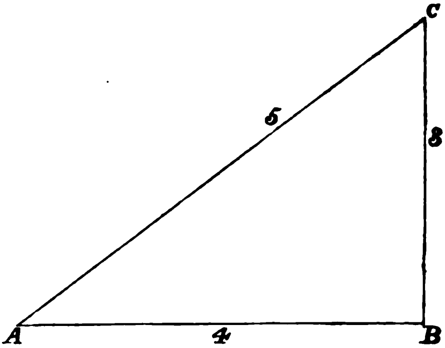
In penning so many sheets, why do you not omit all personal address—and take advantage of the book-post ? My replies are sure to be so short that nothing would be gained by this method, but if I had to send you 18 sheets, I would merely put my Paper in the form of a treatise (not addressing you or any body), just as if I were compiling MSS. for the press for publication.

I will not imitate the scarcely civil terseness of Mr. Wilson, in his note to you, but I do respectfully remind you, that you continually *assume* what you want to prove, that scores of sheets become to me unavailing ; because, I cannot agree with the earlier parts of them.

Thus, in your Letter of 10th August (whose avowed object is to prove a certain angle 36° 52′) you actually assume, in the 3rd sheet, that Sin. 36° 52′ = ·6, without any proof whatever. This is the very thing I deny: as I did in my former correspondence.

If the sides of a right-angled triangle are in ratio 3, 4, 5. Sin. A = $\frac{3}{5}$ = ·6. Cos. A = $\frac{4}{5}$ = ·8.

But this does not shew that A. = 36° 52′. The Sine 36° 52′ is to be computed independently, and this has been done, the result is very nearly ·6, but not quite. Why did you fix on 36° 52′ rather than 36° 53′?



Yours very truly,
GEO. B. GIBBONS.

J. S. ESQ.

My Letter of the 10th August to this gentleman, contained a copy of the enclosed diagram, *(See Diagram XX.)*, and I gave him the same proofs I have given you, that the angle C which is common to the three similar right-angled triangles A B C, *m n* C, and E F C, is an angle of 36° 52′, and its sine ·6.

The following is a copy of my reply, to his favour of the 15th August:—

BRITISH ASSOCIATION,
NORWICH, *22nd August*, 1868.

MY DEAR SIR,

Your Letter of the 15th instant has been forwarded to me from home. At the time of writing it, mine of the 14th instant could not have reached you.

You observe:—" *If Log.-sin of ϕ be given (say* 9·7781186*), this gives ϕ only approximately, and can do no more. And what is more wonderful, you employ Logarithms which you declare to be erroneous, and which I allow to be inexact, to get out exact values!*" When, or where, my dear Sir, have I declared Logarithms to be erroneous? So far from this, in our former Correspondence, I proved many things by Logarithms, and with none of my proofs did you ever attempt to grapple.

Now, my dear Sir, if in reading,—or I should rather say—if in attempting to read my Letter of the 10th instant, you were brought to a "*dead stand*" at the third sheet, and so read no further, that I can't help. All I can say in reply is, that the Logarithmic computations in the first half of my Letter, were given to prove the ratio of side to side in certain right-angled triangles; and so far as the proofs of these ratios are concerned, I might have omitted all allusions to angles of 36° 52′ and 53° 8′. I required the right angle—and the right angle only—to prove the ratio by Logarithm; but, had you read my Letter through, you would have discovered that I employ Hutton's Tables to prove that the angles in question are angles of 36° 52′ and 53° 8′; and yet, that the sines of these angles are ·6 and ·8, and *not* ·5999549 and ·8000338, as Hutton gives them. In this way I have proved that Hutton "*upsets*" himself.

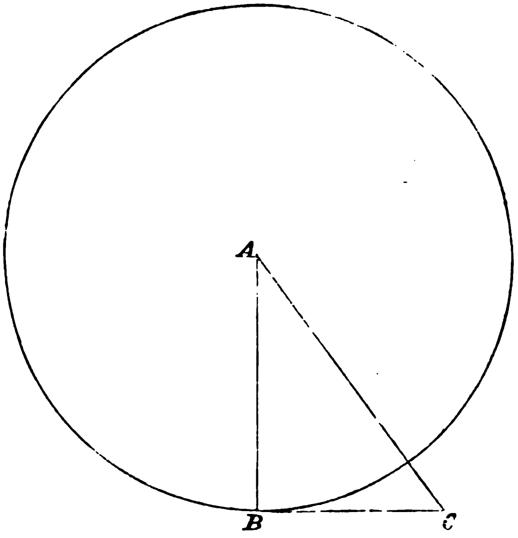
If I get no reply to this, I shall assume that your opinion is unchanged by my reasoning, and on my return home, write you one more Letter, in which I shall demonstrate the true ratio of diameter to circumference in a circle by means of angles, and with this I think our correspondence on the subject may terminate.

Believe me, my dear Sir,

Very truly yours,

JAMES SMITH.

I did not fulfil the promise made at the close of my Letter of the 22nd August, but I wrote to my Correspondent an explanation, and I quote the following from that communication:—*I felt, on consideration, that if the third page of my communication of the* 10*th August brought you to a "dead stand," so that you probably read no further; or, assuming you to have read without being able to understand the proofs I gave you on pages* 14 *to* 18, *that the sines of angles of* 36° 52' *and* 53° 8' *are* ·6 *and* ·8, *and* not ·5999549 *and* ·8000338, *as given in Hutton's Tables, it would be a waste both of your time and mine to give you further trouble.*" The Letter concluded by informing my Correspondent that, while at Norwich, I had the opportunity of telling his friend Professor Adams, that I should give to the world, before the next meeting of the British Association, a demonstration of the true ratio of diameter to circumference in a circle, by means of angles. With this our correspondence terminated.



In the geometrical figure, in the margin, let A B C be a right-angled triangle, of which the sides BC and A B, which contain the right angle, are in the ratio of 3 to 4; and A B the radius of the circle; by construction.

By hypothesis, let A B = $\sqrt{41}$. Then: $\frac{3}{4}$ (A B) $= \frac{3}{4}(\sqrt{41}) = \sqrt{\left(\frac{3^2}{4^2} \times 41\right)} = \sqrt{23{\cdot}0625} =$ B C: and, $\frac{5}{4}$ (A B) $= \frac{5}{4}(\sqrt{41}) = \sqrt{\left(\frac{5^2}{4^2} \times 41\right)} = \sqrt{64{\cdot}0625} =$ A C; therefore, $AB^2 + BC^2 = (\sqrt{41}^2 + \sqrt{23{\cdot}0625}^2) = (41 + 230625) =$

$(41 + 23\cdot0625) = 64\cdot0625 = AC^2$; therefore, $(AB^2 + BC^2 + AC^2) = 3\frac{1}{8}(AB^2)$; that is, $(41 + 23\cdot0625 + 64\cdot0625) = (3\cdot125 \times 41) = 128\cdot125$. Hence: The equation or identity $(AB^2 + BC^2 + AC^2) = 3\frac{1}{8}(AB^2)$ = area of a square, plus the difference between the area of an inscribed circle and the area of an inscribed square to the circle, when the diameter of the circle = 10; that is to say, this equation or identity $= 10^2 + \{(3\frac{1}{8}(5^2) - 50\}$; $= 100 + 28\cdot125 = 128\cdot125$. Again: When $AB = \sqrt{41}$, the equation or identity $(AB^2 + BC^2 + AC^2) = 3\frac{1}{8}(AB^2)$, = the sum of the areas of circles of which the diameters are 8 and 10; that is, $\{3\frac{1}{8}(4^2) + 3\frac{1}{8}(5^2)\} = 3\frac{1}{8}(\sqrt{41})^2$; or, $(50 + 78\cdot125) = (3\cdot125 \times 41) = 128\cdot125$. Again: $4(AB^2) = 4(\sqrt{41})^2 = (4 \times 41) = 164$, = the sum of the areas of circumscribing squares to circles of which the diameters are 8 and 10. Now, by hypothesis, let the arithmetical value of π be either greater or less than $3\cdot125$. Can Professor de Morgan, or any other "*recognised Mathematician*," shew me how, with any such value of π,—whether determinate or indeterminate—we can get the two former equations? The last equation is a truism into which π does not enter; but it will suggest much to any Mathematlcian who possesses the "*two eyes of exact science*," and knows how to make a right use of them.

In the next place, let the diameter of the circle in the figure on page 367 = 10. Then: $AB = 5$, and $\{2(AB)\}^2 \times \frac{\pi}{4}$ = area of the circle, whatever be the value of π. On the hypothesis that 8 circumferences of a circle are exactly equal to 25 diameters, $\frac{25}{8} = 3\cdot125$ is the arithmetical value of π. One of my correspondents—a "*recognised Mathematician*"—on my telling him that I assumed as a THEORY that 8 circumferences = 25 diameters, in every circle; and that I could demonstrate the truth of the *theory* in a hundred ways, was very severe with me. He would have it that "there are *theorems* but no *theories* in Geometry." I quote the following from one of the communications I received from the gentleman referred to in the early part of "*Euclid at Fault.*" "*Unless first principles are well established—proved beyond ques-*

tion, if not axiomatic or self-evident—no discussion can be worth anything or possess the least interest to sincere and intelligent men. I am perfectly sure that Mr. Smith will admit, that if π *cannot be shewn to be a determinate quantity, and shewn by* a priori *reasoning, that is, without reference to its arithmetical value—that process of reasoning by which he some time ago said he arrived at his conviction that* $\pi = 3\frac{1}{8}$ *cannot be valid. That* π *is determinate is, I say, a first principle in that process. Now, I again question the truth of that principle—or rather, proposition. As frequently I have said, it is not self-evident; and I know of no way in which it can be proved* a priori. *Surely it cannot be proved by practical geometry, or by calculations. It must be established by some kind of* a priori *and abstract reasoning: because, it is brought in by Mr. Smith to find the arithmetical value of* π. *Now, I humbly submit that all Mr. Smith can say is away from the point, until he meet this claim I again make, viz., that he shew how we must believe* π *to be a determinate quantity."** Is not reasoning such as this—if reasoning it can be called—equivalent to telling me, that I must find the arithmetical value of π without the aid of arithmetic? How have "*recognised mathematicians*" discovered—as they think—that $\pi = 3{\cdot}14159265$ with a never-ending string of decimals? Have they ever proved *a priori* that the only way of arriving at the ratio of diameter to circumference in a circle, is by polygons? Have they ever proved *a priori* that the only way of arriving at the area of a circle of radius 1, is by circumscribed and inscribed polygons to a circle?

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* In the short address to the Reader, I have proved that π must be a *finite and determinate* arithmetical quantity, by *a priori* or abstract reasoning: and I have proved elsewhere (*see page* 324) that we may assume π to be anything intermediate between 3 and 4, and get the equation or identity, $4\left(\frac{\pi}{2}\right)^2 = \pi^2$. It is simply absurd, for any one to conceive the idea of proving, whether π is, or is not, a *finite and determinate* quantity, "*without reference to its arithmetical value.*" There is a reference to its arithmetical value, in the proof that π must be greater than 3 and less than 4; and also in the proof, that $12\frac{1}{2}\left(\frac{\pi}{50}\right) = \frac{\pi}{4}$ = area of a circle of diameter unity, whatever be the value of π.

Certainly not! They may tell me that these things are self-evident. That I deny. They are not self-evident to me, and I do not believe that either you, or any other living Mathematician, can prove either one or the other, "*by the rules of logic and common sense:*" and in the equations I have already given you, I have proved that π can be nothing else but $3{\cdot}125$. Hence: In the analogy or proportion, $A : B :: B : C$; when $A = \frac{3{\cdot}125}{4}$, and $B = 1$, then, $C = 1{\cdot}28$; that is, $\cdot78125 : 1 :: 1 : 1{\cdot}28$, and it follows, that $\frac{\cdot78125}{1}$ and $\frac{1}{1{\cdot}28}$ are equivalent ratios, and both express the ratio between the area of every circle and the area of its circumscribing square.

Well, then, let the diameter of the circle in the figure on page $367 = 10$. Then: $A B = 5$: and $\{2 (A B)\}^2 \times \frac{\pi}{4} = \frac{\{2(A B)\}^2}{1{\cdot}28}$, that is, $\left(10^2 \times \frac{3{\cdot}125}{4}\right) = \frac{10^2}{1{\cdot}28}$; or, $100 \times \cdot78125 = \frac{100}{1{\cdot}28}$; and this equation $= 3\frac{1}{8} (A B^2) = 3{\cdot}125 (5^2) = 3{\cdot}125 \times 25 = 78{\cdot}125 =$ area of the circle. Again: $\{2 (A B\}^2 = (2 \times 5)^2 = 10^2 = 100 =$ area of a circumscribing square to the circle: and, $\frac{\{2(AB)\}^2}{2} = \frac{100}{2} = 50 =$ area of an inscribed square to the circle. Hence: $\left\{\left(50 + \frac{50}{4}\right) + \frac{1}{4}\left(50 + \frac{50}{4}\right)\right\} = \frac{100}{1{\cdot}28}$; that is, $(62{\cdot}5 + 15{\cdot}625) = \frac{100}{1{\cdot}28}$, and this equation $= 3\frac{1}{8} (A B^2) = 78{\cdot}125 =$ area of the circle. Again: $\{2 (A B)\}^2 - 7(\pi)\} = \{2 (A B)\}^2 \times \frac{\pi}{4}$; that is, $(100 - 21{\cdot}875) = (100 \times \cdot78125)$, and this equation $= 3\frac{1}{8} (A B^2) = 78{\cdot}125 =$ area of the circle. Again: $2 (A B^2) + 9 (\pi) = \frac{\{2 (A B)\}^2}{1{\cdot}28}$; that is, $50 + (9 \times 3{\cdot}125) = \frac{100}{1{\cdot}28}$, or, $(50 + 28{\cdot}125) = \frac{100}{1{\cdot}28}$, and this equation $= 3\frac{1}{8} (A B^2) = 78{\cdot}125 =$ area of the circle. Lastly: $\{2 (A B)^2 - 7 (\pi)\} = 2 (A B^2) + 9 (\pi)$; that is, $(100 - 21{\cdot}875) = (50 + 28{\cdot}125)$, and this equation $= 3\frac{1}{8} (A B^2) = 78{\cdot}125 =$ area of the circle.

Once more: $12\frac{1}{2}$ times $\frac{\pi}{50} =$ area of a circle of diameter unity, whatever be the value of π; and it follows of necessity. that $12\frac{1}{2}$

times the area of a square on the semi-radius, = area in every circle. For example: Let the semi-radius of a circle = $\sqrt{60}$. Then: the area of a square on the semi-radius = $\sqrt{60}^{2} = 60$: and, $12\frac{1}{2}(60) = 12{\cdot}5 \times 60 = 750$ = area of the circle. Proof: $\frac{750}{\frac{1}{4}(\pi)} = 750 \times 1{\cdot}28$; that is, $\frac{750}{{\cdot}78125} = 750 \times 1{\cdot}28$, and this equation = 960 = area of a circumscribing square to the circle; and it follows, that $\sqrt{960}$ is the diameter of the circle. But, $\frac{\text{Diameter}}{2} = \frac{\sqrt{960}}{2} = \sqrt{240}$ = radius of the circle; and $\pi(r^2)$ = area in every circle; therefore, $\pi(\sqrt{240}^{2}) = 3{\cdot}125 \times 240 = 750$ = area of the circle. Q. E. D.

Now, let A B the radius of the circle, in the geometrical figure on page 367 = $\sqrt{240}$. Then: $\frac{3}{4}(A\,B) = \frac{3}{4}\left(\sqrt{240}\right) = \sqrt{\left(\frac{3^2}{4^2} \times 240\right)} = \sqrt{135} = B\,C$: and, $\frac{5}{4}\left(A\,B\right) = \frac{5}{4}\left(\sqrt{240}\right) = \sqrt{\left(\frac{5^2}{4^2} \times 240\right)} = \sqrt{375} = A\,C$; therefore, $(A\,B^2 + B\,C^2 + A\,C^2) = 3\frac{1}{8}(A\,B^2)$; that is, $(240 + 135 + 375) = (3{\cdot}125 \times 240)$, and this equation = 750 = area of the circle.

When I find that truths like the foregoing cannot find an entrance into the minds of "*recognised Mathematicians*," I am tempted to put the question:—Is it an effect of "*crammed erudition*" in the science of Mathematics, to darken the understanding? So far as my experience goes of *professional* "*recognised Mathematicians*"—and it is not a little—it would appear to be so, and I am not now surprised at anything they say; indeed, it would not now surprise me, if I found them asserting that, the Multiplication Table is a *mockery*, *delusion*, and *a snare:* Addition a *hindrance* and a *pitfall:* Subtraction a *shoal* and a *shallow:* Division a *snake in the grass:* and Arithmetical proportion a combination of all.

Referring you to the enclosed diagram *(see Diagram XX.)*, you will observe, that every triangle in the figure is connected with the straight line F C; and yet, that while B C and *n* C, parts of it, are radii of the circles Y and Z, F C is not itself a radius of any of the circles, Let BC, the radius of the circle Y, $= 4$. Then: $AB = 3$: $AO = 1$: $OD = 1$: $AD = 2$: $mn = 3{\cdot}75$: $EF = 3{\cdot}8418744$: $nB = 1$: $FB = 1{\cdot}1224992$: $FC = 5{\cdot}1224992$: $AC = nC = 5$: $mC = 6{\cdot}25$: $CO = \sqrt{32}$: and, $EC = DC = \sqrt{41} = 6{\cdot}403124$.

Now CDB is a right-angled triangle, and CAD and COD, parts of it, are oblique-angled triangles.

Hence:

$CA^2 + AD^2 + 2(AB \times AD) = DC^2$; that is, $5^2 + 2^2 + 2(3 \times 2)$; or, $(25 + 4 + 12) = 41 = DC^2$.

$CO^2 + OD^2 + 2(OD \times OB) = DC^2$; that is, $\sqrt{32}^{\,2} + 1^2 + 2(4 \times 1)$; or, $(32 + 1 + 8) = 41 = DC^2$.

Again: mnC is a right-rngled triangle, and if mB be joined, then mBC, a part of it, will be an oblique-angled triangle: and $mn^2 + nB^2 = 3{\cdot}75^2 + 1^2 = 14{\cdot}0625 + 1 = 15{\cdot}0625 = mB^2$; therefore, $mB = \sqrt{15{\cdot}0625}$.

Hence:

$mB^2 + BC^2 + 2(BC \times Bn) = mC^2$; that is, $\{\sqrt{15{\cdot}0625}^{\,2} + 4^2 + 2(4 \times 1)\}$; or, $(15{\cdot}0625 + 16 + 8) = 39{\cdot}0625 = MC^2$; therefore, $\sqrt{39{\cdot}0625} = 6{\cdot}25 = mC$.

Again:

EFC is a right-angled triangle, and if EB be joined, then EBC, a part of it, will be an oblique-angled triangle; and, $EF^2 + FB^2 = 3{\cdot}8418744^2 + 1{\cdot}1224992^2 = 14{\cdot}759998905 37536 + 1{\cdot}26000445400064 = 16{\cdot}020003359376 = EB^2$.

Hence:

$EB^2 + BC^2 + 2(BC \times BF)$; that is, $(16{\cdot}020003359376 + 16 + 2(4 \times 1{\cdot}1224992)$; or, $(16{\cdot}020003359376 + 16 + 8{\cdot}9799936) = 40{\cdot}999996959376$; and is a very close approximation to EC^2.

You may say, this is an apparent proof that Euclid is not at fault in the 12th Proposition of his second book. This may be granted: and, when you can wield that "*indispensable instrument of science, Arithmetic,*" and—with reference to the diagram in "*Euclid at Fault*"—prove that $HT^2 + TB^2 + 2(TB \times TP) = HB^2$, you will demonstrate that the proposition in question, is "*of general and universal application,*" and *true* "*under all circumstances*" If, as you say, you do not "*despise instruction,*" you will take these facts in connection with my Letter of the 27th October, *test them "by the rules of logic and common sense,"* and if so, you cannot fail to arrive at a true conclusion.

This epistle has run to a length far beyond what I anticipated, but I hope I shall be able to keep future communications within moderate compass.

I remain, dear Sir,
Yours very respectfully,
JAMES SMITH.

J. M. WILSON, ESQ.

JAMES SMITH *to* J. M. WILSON, ESQ.

BARKELEY HOUSE, SEAFORTH,
16th November, 1868.

DEAR SIR,

You are—or at any rate might be—aware of the fact, that my Letter to His Grace the Duke of Buccleuch, brought me—through the intervention of a friend—into communication with a "*recognised Mathematician.*" The correspondence was a long one, and I was about to publish it, and gave my opponent the opportunity of revising his own Papers, which he did so far as his early communications were concerned, without altering a word. But, subsequently, finding he had got into a rather perplexing position, he wished, and attempted, to make such alterations in his communications, as would have resolved my replies into perfect nonsense. This I could not submit to, but out of deference to the wishes of my friend, who was the medium between us, I have withheld this correspondence from the public, although 234 pages of it are actually printed off. It was this circumstance that led me to throw off, *very hastily*, the pamphlet "*Euclid at Fault.*"

PROBLEM.

Construct a geometrical figure, in which there shall be—isolated and exhibited—an isosceles triangle, of which one-tenth part of the arithmetical value of the base, is the sine of half the angle at the apex: and also contain—isolated and exhibited—a circle and a square of exactly the same superficial area.

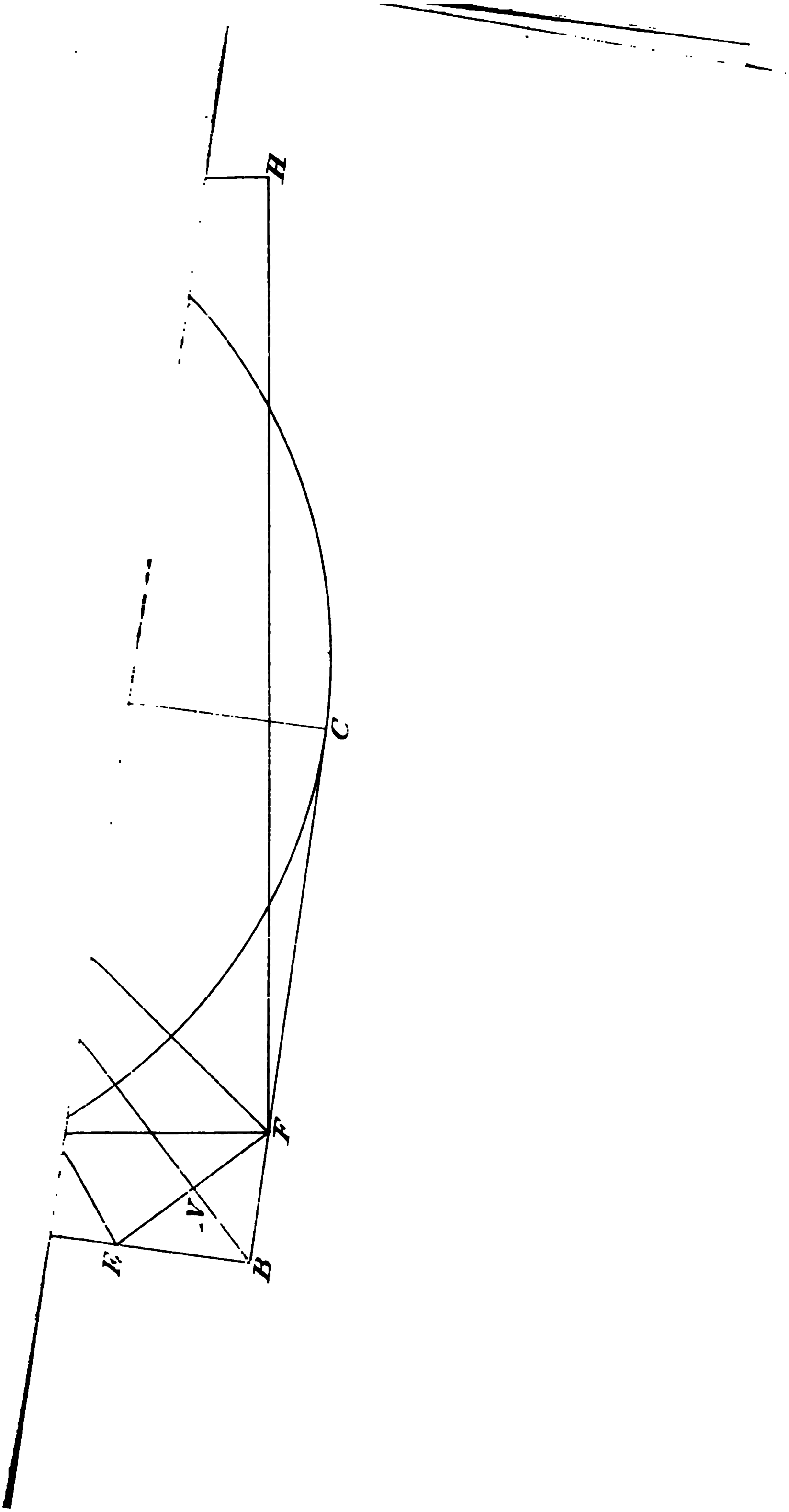
The enclosed Diagram *(See Diagram XXI.)* solves this problem.

CONSTRUCTION.

On the straight line A B describe the square A B C D, and with D as centre and D A or D C as interval, describe the circle. From B A cut off a part B E equal to one-fourth part of B A; and from B C cut off a part B F equal to one-fourth part of B C, and join E F. Produce B A to G, making A G equal to A E, and join G F, D E, D B, and D F. On G F describe the square G F H K.

Now, by analogy or proportion, G B : B F : : D N : N B, but this fact cannot be demonstrated by pure Geometry. We can only furnish the proof by *applied Mathematics;* or in other words, we can only get at the proof by "*wielding that indispensable instrument of Science, Arithmetic.*"

Well, then, by hypothesis, let A B = 4. Then: E B = 1 : A G = A E = 3; therefore, (A G + A E + E B) = (3 + 3 + 1) = 7 = G B. B F = B E = 1, and E B F is a right-angled isosceles triangle; therefore, $(EB^2 + BF^2) = (1^2 + 1^2) = (1 + 1) = 2 = EF^2$; therefore, $EF = \sqrt{2}.$ But, E F is bisected at N, and the isosceles triangle E B F is divided by the line B N into two similar and equal right-angled isosceles triangles, and it follows, that N E, N B, and N F are equal, $= \frac{\sqrt{2}}{2} = \sqrt{\cdot 5}.$ But, D A = A B = 4: and A E = 3, and D A E is a right-angled triangle; therefore, $(DA^2 + AE^2) = (4^2 + 3^2) = (16 + 9) = 25 = DE^2$; therefore, $\sqrt{25} = 5 = DE.$ But, D N E is a right-angled triangle; therefore, $(DE^2 - NE^2) = (5^2 - \sqrt{\cdot 5}^{\,2}) = (25 - \cdot 5) = 24\cdot 5 = DN^2$; therefore, $DN = \sqrt{24\cdot 5}.$ But, B F = B E = 1; therefore, 7(B F) = (7 × 1) = 7 = G B; and $7\,(NB) = 7\,(NE) = 7\,(\sqrt{\cdot 5}) = \sqrt{7^2 \times \cdot 5} = \sqrt{49 \times \cdot 5} = \sqrt{24\cdot 5} = DN$; and it follows, that G B F, D N E, and D N F are similar right-angled triangles, and have the sides that contain the right angle in the ratio of 7 to 1, Hence: $\frac{EN}{ED} = \frac{EF^2}{10}$, that is,


H
C
F
V
E
B

$\frac{\sqrt{\cdot 5}}{5} = \frac{\sqrt{2}}{10}$, or, $\frac{\cdot 7071068}{5} = \frac{1\cdot 414213}{10}$, and this equation = ·1414213: and ·1414213 is the arithmetical value of the sine of an angle of 8° 8′ to seven places of decimals. These facts can be demonstrated by means of Logarithms.

I have proved that when A B = 4, D N = $\sqrt{24\cdot 5}$; N E = $\sqrt{\cdot 5}$: and D E = 5. Now, $\sqrt{24\cdot 5}$ = 4·949747 = the side D N in the right-angle triangle D N E: and $\sqrt{\cdot 5}$ = ·7071068 = the side N E. But, $\frac{\text{N E}}{\text{D E}} = \frac{\cdot 7071068}{5}$ = ·1414213, and ·1414213 is the sine of the angle NDE: and, $\frac{\text{D N}}{\text{D E}} = \frac{4\cdot 949747}{5}$ = ·9899494, and ·9899494 is the sine of the angle D E N. The Logarithm corresponding to the natural number ·1414213 is 9·1505148, and this is the Log.-sin. of the angle N D E. The Logarithm corresponding to the natural number ·9899494 is 9·9956129, and this is the Log.-sin. of the angle D E N.

Let D E, the side subtending the right angle, in the triangle D N E, be any given length, say 90 miles, and be given to find the lengths of the other two sides N E and D N, and prove that they are in the ratio of 7 to 1. Then: The angle N D E is an angle of 8° 8′; therefore, 90° — 8° 8′ = 81° 52′ = the angle D E N.

Then:

As Sin. of angle N = Sin. 90° ....Log. 10·0000000
: the given side D E = 90 miles ....Log. 1·9542425
: : Sin. of angle N D E = Sin. 8° 8′ ....Log. 9·1505148

11·1047573
10·0000000

: the side N E =
90 × ·1414213 = 12·727917 miles.....................Log. 1·1047573

Again:

As Sin. of angle N = Sin. 90° ....Log. 10·0000000
: the given side D E = 90 miles ....Log. 1·9542425
: : Sin. of angle D E N = Sin. 81° 52′ ....Log. 9·9956129

11·9498554
10·0000000

: the side D N =
90 × ·9899494 = 89·095446 miles.....................Log. 1·9498554

Hence: 7(N E) = D N ; that is, 7 × 12·727917 = 89·095419, correct to the fourth place of decimals.

By Hutton's Tables:

As Sin. of angle N = Sin. 90° ....Log. 10·0000000
: the given side D E = 90 miles ....Log. 1·9542425
: : Sin. of angle N D E = Sin. 8° 8′ ....Log. 9·1506864

11·1049289
10·0000000

: the side N E =
90 × ·1414772 = 12·732948 miles ....Log. 1·1049289

Again:

As Sin. of angle N = Sin. 90° ....Log. 10·0000000
: the given side D E = 90 miles ....Log. 1·9542425
: : Sin. of angle N D E = Sin. 81° 52′ ....Log. 9·9956095

11·9498520
10·0000000

: the side D N =
90 × ·9899415 = 89·094735 miles......................Log. 1·9498520

But this makes 7 (N E) greater than D N, that is to say, 7 × 12·732948 = 89·130596, and is greater than 89·094735, the length of D N, as ascertained by Hutton's Tables; which destroys the ratio—a known and indisputable ratio by the construction of the figure—between the sides N E and D N, in the right-angled triangle D N E. How, then, "*by the rules of logic and common sense,*" are Mathematicians to get over the fact, that Mathematical Tables of Sines, Cosines, &c., are fallacious?

Well, then, $GF^2 = (DC^2 + CF^2 + DF^2)$, and this equation = area of the square G F H K. But, $(DC^2 + CF^2 + DF^2) = 3\frac{1}{8}(DC^2)$, and this equation = area of the square G F H K. But, $3\frac{1}{8}(DC^2) = (GB^2 + BF^2)$, and this equation = area of the square G F H K. But, $GB^2 + BF^2 = 3\frac{1}{8}\left\{(GB + BF) \times \frac{DC}{2},\right\}$ and this equation = area of the square G F H K. Hence: Since πr^2 = circumference × semi-radius, and since this equation = area in every circle, it follows of necessity—unless Mathematicians can find some other value of π than $3\frac{1}{8}$, which, multiplied by D C^2, will give the area of the square G F H K—that the circle and the square G F H K are exactly of the same superficial area.

Now, the angles E and F, in the right-angled isosceles triangle E B F, are angles of 45°, and are together = to the right-angle B; and the angle B is itself divided into two angles of 45°. The angle B G F + the angle F D C = (8° 8′ + 36° 52′), and are together = to half a right angle = 45°. The four angles C D F, F D N, N D E, and E D A, = (36° 52′ + 8° 8′ + 8° 8′ + 36° 52′), and are together equal to the right angle A D C = 90°. The four angles A E D, D E N, D F N, and D F C = (53° 8′ + 81° 52 + 81° 52′ + 53° 8′), and are together equal to three right angles = 270°. The angles A D E, A E D, E D N, and D E N = (36° 52′ + 53° 8′ + 8° 8′ + 81° 52′), and are together = to two right angles; and it follows of necessity, that the eight angles A D E, A E D, E D N, D E N, N D F, D F N, D F C, and F D C, are together equal to four right angles; and, supposing A D and C D to be produced to meet the circumference of the circle, and so, producing four right angles at the centre of the circle. It is self-evident, that these eight angles are together equal to the four angles at the centre of the circle.

You may tell me that the latter fact is true, whatever be the value of the angles. *Granted!* But the *onus-probandi* rests with Mathematicians *to prove*, that the values of the angles, as given above, are false values.

Nov. 19.

I had written so much of this Letter on the day of its date, but had a reason for not finishing and posting it.

This morning's post brought me your letter of yesterday's date, upon which I make no comments.

I am, Sir,

Yours respectfully,

JAMES SMITH.

J. M. WILSON, Esq.

P.S.—You would probably prefer not to see any more of my "*hand-writing.*" If so, and you will be pleased to tell me, it may save both of us some trouble, and me something in "*postage stamps.*" I will present you with a copy of my work when it comes out ; and it may be, that you would prefer to work out "*The Geometry of the Circle, and the applications of proportion to Plane Geometry,*" in your own way, without further interruption from me. If this be your wish, please intimate it. You will, of course understand that I shall make no change in the form in which I intend to publish my work ; that is to say, it will come out in the form of a series of Letters to you, with your replies, which have never been marked *private.**

J. S.

We *know* that 6 (radius × semi-radius) = area of a regular inscribed dodecagon, to every circle.

Well, then, let D C a radius of the circle in the geometrical figure represented by Diagram XXI. = $\sqrt{2}$: and let A denote the area of a regular dodecagon inscribed in the circle.

Then :

$$6\left(DC \times \frac{DC}{2}\right) = 6(\sqrt{2} \times \sqrt{\cdot 5}) = 6(\sqrt{1}). = 6 = A.$$

$$\text{Proof: } 6(\sqrt{1}) = \sqrt{6^2 \times 1} = \sqrt{36 \times 1} = \sqrt{36} = 6 = A.$$

The fact is indisputable, that $\frac{3}{\pi}$ expresses the ratio between the perimeter of every regular hexagon and the circumference of its circumscribing circle, whatever be the value of π. Will any Mathematician attempt to controvert this fact ? I trow not! Might he not as well attempt to prove that 6 times radius is *not* equal to the perimeter of any regular inscribed hexagon to a circle ? Is not 6 times radius = the perimeter of a regular inscribed hexagon, to every circle? Well, then, it follows of necessity, that $\frac{3}{\pi}$ expresses the ratio between the area of every regular dodecagon, and the area of its circumscribing circle.

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* The reader can hardly fail to perceive, that I was forced into a diversion from this intention, by the "*recognised Mathematician,*" the Rev. Professor Whitworth.

Hence:

$3 : \pi :: A : 2\pi$, when the radius of the circle $= \sqrt{2}$, whatever be the value of π.

Let the radius of the circle $= 2$. Then: $3 : \pi :: A : 4\pi$, whatever be the value of π. Again: Let the radius of the circle $= 4$. Then: $3 : \pi :: A : 16\pi$, whatever be the value of π. Again: Let the radius of the circle $= 8$. Then: $3 : \pi :: A : 64\pi$, whatever be the value of π. Again: Let the radius of the circle $= 16$. Then: $3 : \pi :: A : 256\pi$, whatever be the value of π.

For example:

Let D C a radius of the circle $= 16$: Then: $6\left(DC \times \frac{DC}{2}\right) = 6(16 \times 8) = (6 \times 128) = 768 = A$; therefore, $3 : \pi :: A : 256(\pi)$ whatever be the value of π. By hypothesis, let $\pi = 3{\cdot}1416$. Then: $3 : 3{\cdot}1416 :: 768 : 256(3{\cdot}1416)$; that is, $3 : 3{\cdot}1416 :: 768 : 804{\cdot}2496$. Again: By hypothesis, let $\pi = \frac{25}{8} = 3{\cdot}125$. Then: $3 : \pi :: 768 : 256(3{\cdot}125)$, that is, $3 : 3{\cdot}125 :: 768 : 800$. On both hypotheses, $\pi(r^2) =$ area of the circle.

It matters not whether we hypothetically assume $\pi = 3$, or $\pi = 4$, or $\pi =$ any *finite and determinate* arithmetical quantity, intermediate between 3 and 4: on any hypothesis, we shall arrive at the same conclusion; that is to say, we shall demonstrate that $\pi(r^2) =$ area in every circle, whatever be the value of π. It may be admitted that this proves nothing as to the true arithmetical value of π; but, will any *honest* Mathematician attempt to controvert the fact, that it demonstrates beyond the possibility of dispute or cavil, that whatever be the arithmetical value of the symbol π, it cannot be an indeterminate arithmetical quantity ? Does not that "*recognised Mathematician*," the Rev. Professor Whitworth, make the following assertion :—"*I know, and always teach, that the value of π is a finite and determinate quantity.*" (See the Professor's Letter of Nov. 9, 1868, page 2). Will the learned Professor be good enough to inform us, if he differs from those Mathematicians, who assign to the symbol π the indeterminate arithmetical quantity 3·14159265, &c. ?

Now, 2π (radius) = circumference in every circle, and $\pi(r^2)$ = area in every circle. But, $\pi(r^2)$ = circumference × semi-radius, and it follows, that this equation = area in every circle. Let D C the radius of the circle = 16, and by hypothesis, let $\pi = 3{\cdot}1416$. Then : $2\pi(\text{D C}) = 2\pi(16) = (6{\cdot}2832 \times 16) = 100{\cdot}5312$ = circumference; and $\frac{\text{D C}}{2} = \frac{16}{2} = 8$ = semi-radius ; therefore, circumference × semi-radius = $(100{\cdot}5312 \times 8) = \pi(\text{D C}^2)$; that is, $(100{\cdot}5312 \times 8) = 3{\cdot}1416\,(256)$, and this equation = 804·2496 = area of the circle, on the hypothesis that $\pi = 3{\cdot}1416$. But, any other hypothetical value of π, intermediate between 3 and 4, will produce a similar result, so that it be *finite and determinate.* How, then, is it possible that π can be an *indeterminate* arithmetical quantity ? It should not be necessary for "*recognised Mathematicians*," such as De Morgan, Wilson, and Whitworth, to force upon me the necessity of teaching *them* such plain and simple geometrical and mathematical truths as these !

J. S.

J. M. WILSON, ESQ. *to* JAMES SMITH.

RUGBY, *November* 16*th*, 1868.

MY DEAR SIR,

I have received your two somewhat extended Letters, and beg to acknowledge them. I cannot say that I have perused them with an attention at all proportionate to their length ; but I have been able to perceive that you are able to prove your conclusions by a process of reasoning which is absolutely sound and logical, and that therefore your conclusions are as certain as their premises, with which they are in fact identical.

This will, I hope, be accepted as my recantation, and be published along with your Letters to me.

Sir, I admire your indomitable perseverance, your hand-writing, and your liberal expenditure of postage stamps. They are worthy of a more extended success than they have yet met with. But pray accept, as an instalment of the debt that will not be fully paid in your life-time, my admission above made. You are obtaining recruits at last from the ranks of professed Mathematicians.

Believe me, respectfully yours,

JAMES M. WILSON.

JAMES SMITH *to* J. M. WILSON, ESQ.

BARKELEY HOUSE, SEAFORTH,
17*th November*, 1868.

MY DEAR SIR,

Actuated simply by a sense of duty, I have for nearly ten years, as opportunities have occurred—and at much personal cost—brought the subject of the true ratio of diameter to circumference in a circle, before the scientific public : cheifly in the form of pamphlets, distributed at the meetings of *the British Association for the Advancement of Science.* In 1859, after very considerable difficulty, I did succeed in reading a short Paper in the Physical and Mathematical Section : the fact—not the Paper—is recorded in the Transactions of the Association for that year ; but I have never been permitted to read another. Abuse and ridicule have been heaped upon me from many quarters, public as well as private, at which I could always afford to smile : and I have never been driven from my onward path:

Now, my dear Sir, after spending so much of apparently fruitless time and labour on this and kindred subjects, you can readily conceive the extreme gratification I felt this morning, on reading your frank and complimentary Letter of yesterday, for which accept my most sincere and hearty thanks. I begin to feel that there are now some spots of sunshine, in what has hitherto been to me, a dark horizon ; and that it is possible I may yet live to see, that my labours in the cause of science, have not been in vain.

In Liverpool I am very well known as a public man ; but, still better, by the sobriquet of "*The Circle Squarer.*" I pass for a man of sound mind on most subjects ; but there is a large class of non-scientific persons in Liverpool, who have a great personal respect for me ; but who, taking their tone from the press, would tell you, that if you touch me on circle squaring, I am as "*mad as a March hare.*" I could get a copy of your Letter inserted in one of the Liverpool Journals, and if I had your permission to do so, this would *at once* put me in my right position, with the class of persons referred to. I ask this favor at your hands ?

Believe me, my dear Sir,

Very truly yours,

JAMES SMITH.

J. M. WILSON, ESQ.

J. M. WILSON, ESQ. *to* JAMES SMITH.

RUGBY, 18*th November*, 1868.

MY DEAR SIR,

I did not think my Letter would have deceived you for more than a moment. I said that your conclusions were as certain *as their premises, with which they are, in fact, identical.*

You do assume the result in the premises, from which you correctly argue.

I entirely withhold my consent from any publication of private Letters of mine.

Once more, let me assure you, that you are wrong in your premises and conclusions: and that I should be particularly gratified if I could hope to persuade you of this.

Yours,

J. M. WILSON.

JAMES M. WILSON, ESQ., *to* JAMES SMITH.

RUGBY, *November 20th*, 1868.

DEAR SIR,

It is certainly my wish, which I expressed before, that you would discontinue your correspondence.

You gave me notice in the first instance that it was your intention to publish your own Letters, and you are at liberty to do so. But you are not at liberty to publish Letters which you may receive, whether marked "*private*" or not, without the consent of the writers. This is not only a convention amongst gentlemen, which you might set aside, but it is law. I now distinctly forbid the publication of any of my Letters, and if you publish them in spite of this notice you will take the consequences. I shall put the matter into the hands of my Solicitor.

This Letter closes the correspondence on my part.

Truly yours,

JAMES M. WILSON.

JAMES SMITH *to* J. M. WILSON, ESQ.

BARKELEY HOUSE, SEAFORTH,
21st November, 1868.

SIR,

When you recanted the pretended "*recantation*" that imposed upon my credulity, should you not, according to the "*conventions amongst gentlemen, have pointed out the fallacy in my premises, by which you are led into the belief of the fancy*," that I am wrong in my conclusions.

When you penned your last communication, you must surely have forgotten the following paragraph, which I quote from your Letter of the 16th instant:—"*This will I hope be accepted as my recantation, and be published along with your Letters to me.*"

As regards your threat to take legal proceedings, I shall not hesitate to pursue my course, already indicated—why should I ?—and leave you to pursue yours.

I shall give you no further trouble, until I present you with a copy of my next publication, which will afford you the opportunity of taking legal proceedings, if you consider it your interest to do so.

Yours faithfully,

JAMES SMITH.

J. M. WILSON, ESQ.

APPENDIX D.

FROM THE "ATHENÆUM," JANUARY 4TH, 1868.

PSEUDOMATH, PHILOMATH, AND GRAPHOMATH.

December 31, 1867.

MANY thanks for the present of Mr. James Smith's letters of Sept. 28 and of Oct. 10 and 12. He asks where you will be if you read and digest his letters: you probably will be somewhere first. He afterwards asks what the WE of the *Athenæum* will be if, finding it impossible to controvert, it should refuse to print. I answer for you, that We-We of the *Athenæum*, not being Wa-Wa the wild goose, so conspicuous in 'Hiawatha,' will leave what controverts itself to print itself, if it please.

Philomath is a good old word, easier to write and speak than *mathematician*. It wants the words between which I have placed it. They are not well formed; *psuedomathete* and *graphomathete* would be better: but they will do. I give an instance of each.

The *pseudomath* is a person who handles mathematics as the monkey handled the razor. The creature tried to shave himself as he had seen his master do; but, not having any notion of the angle at which the razor was to be held, he cut his own throat. He never tried a second time, poor animal! but the pseudomath keeps on at his work, proclaims himself clean-shaved, and all the rest of the world hairy. So great is the difference between moral and physical phenomena! Mr. James Smith is, beyond doubt, the great pseudomath of our time. His $3\frac{1}{8}$ is the least of a wonderful chain of discoveries. His books, like Whitbread's barrels, will one day reach from Simpkin & Marshall's to Kew, placed upright, or to Windsor laid lengthways. The Queen will run away on their near approach, as Bishop Hatto did from the rats: but Mr. James Smith will follow her were it to John o' Groats.

The philomath, for my present purpose, must be exhibited as giving a lesson to presumption. The following anecdote is found in Thiébault's 'Souvenirs de vingt ans de séjour à Berlin,' published in 1804. The book itself got a high character for truth. In 1807 Marshal Mollendorff answered an inquiry of the Duc de Bassano, by saying that it was the most veracious of books, written by the most honest of men. Thiébault does not claim personal knowledge of the anecdote, but he vouches for its being received as true all over the north of Europe.

Diderot paid a visit to Russia at the invitation of Catherine the Second. At that time he was an atheist, or at least talked atheism: it would be easy to prove him either one thing or the other from his writings. His lively sallies on this subject much amused the Empress, and all the younger part of her Court. But some of the older courtiers suggested that it was hardly prudent to allow such unreserved exhibitions. The Empress thought so too, but did not like to muzzle her guest by an express prohibition: so a plot was contrived. The scorner was informed that an eminent mathematician had an algebraical proof of the existence of God, which he would communicate before the whole Court, if agreeable. Diderot gladly consented. The mathematician, who is not named, was Euler. He came to Diderot with the gravest air, and in a tone of perfect conviction said, *Monsieur!*

$$\frac{a+b^n}{n}=x$$

donc Dieu existe; répondez! Diderot, to whom algebra was Hebrew —though this is expressed in a very roundabout way by Thiébault— and whom we may suppose to have expected some verbal argument of alleged algebraical closeness, was disconcerted; while peals of laughter sounded on all sides. Next day he asked permission to return to France, which was granted. An algebraist would have turned the tables completely by saying, "Monsieur! vous savez bien que votre raisonnement demande le développement de x suivant les puissances entières de n." Goldsmith could not have seen the anecdote, or he might have been supposed to have drawn from it a hint as to the way in which the Squire demolished poor Moses.

The *graphomath* is a person who, having no mathematics, attempts to describe a mathematician. Novelists perform in this way: even Walter Scott now and then burns his fingers. His dreaming calculator, Davy Ramsay, swears "by the bones of the immortal Napier." Scott thought that the philomaths worshipped relics: so they do, in one sense. Look into Hutton's Dictionary for *Napier's Bones*, and you shall learn all about the little knick-knacks by which he did multiplication and division. But never a bone of his own did he contribute; he preferred elephants' tusks. The author of 'Headlong Hall' makes a grand error, which is quite high science: he says that Laplace proved the precession of the equinoxes to be a periodical inequality. He should have said the variation of the obliquity. But the finest instance is the following:—Mr. Warren, in his well-wrought tale of the martyr-philosopher, was incautious enough to invent the symbols by which his *savant* satisfied himself Laplace was right on a doubtful point. And this is what he put together—

$$\sqrt{-3}\,a^4,\ \Box\,\frac{y^2}{z^2} + 9 - n = 9,\ n \times \log e.$$

Now, to Diderot and the mass of mankind this might be Laplace all over: and, in a forged note of Pascal, would prove him quite up to gravitation. But I know of nothing like it except in the lately revived story of the American orator, who was called on for some Latin, and perorated thus:—"Committing the destiny of the country to your hands, Gentlemen, I may without fear declare, in the language of the noble Roman poet,

> E pluribus unum,
> Multum in parvo,
> Ultima Thule,
> Sine qua non."

But the American got nearer to Horace than the martyr-philosopher to Laplace. For all the words are in Horace, except *Thule*, which might have been there. But $='$ is not a symbol wanted by Laplace; nor can we see how it could have been: in fact, it is not recognized in algebra. As to the junctions, &c., Laplace and Horace are about equally well imitated.

Further thanks for Mr. Smith's letters to you of Oct. 15, 18, 19, 28, and Nov. 4, 15. The last of these letters has two curious discoveries. First, Mr. Smith declares that he has *seen* the editor of the *Athenæum*: in several previous letters he mentions a name. If he knew a little of journalism he would be aware that editors are a peculiar race, obtained by natural selection. They are never seen, even by their officials; only heard down a pipe. Secondly, an "ellipse or oval" is composed of four arcs of circles. Mr. Smith has got hold of the construction I was taught, when a boy, for a pretty four-arc oval. But my teachers knew better than to call it an ellipse: Mr. Smith does not; but he produces from it such confirmation of $3\frac{1}{8}$ as would convince any *honest* editor.

Surely the cylometer is a Darwinite development of a spider, who is always at circles, and always begins again when his web is brushed away. He informs you that he has been privileged to discover truths unknown to the scientific world. This we know; but he proceeds to show that he is equally fortunate in art. He goes on to say that he will make use of you to bring those truths to light, "just as an artist makes use of a dummy for the purpose of arranging his drapery." The painter's lay-figure is for flowing robes; the hairdresser's dummy is for curly locks. Mr. James Smith should read Sam Weller's pathetic story of the "four wax dummies." As to *his* use of a dummy, it is quite correct. When I was at University College, I walked one day into a room in which my Latin colleague was examining. One of the questions was, "Give the lives and fates of Sp. Mælius and Sp. Cassius." Umph! said I, surely all know that Spurius Mælius was whipped for adulterating flour, and that Spurius Cassius was hanged for passing bad money. Now, a robe arranged on a dummy would look just like the toga of Cassius on the gallows. Accordingly, Mr. Smith is right in the drapery-hanger which he has chosen: he has been detected in the attempt to pass bad circles. He complains bitterly that his geometry, instead of being read and understood by you, is handed over to me to be treated after my scurrilous fashion. It is clear enough that he would rather be handled in this way than not handled at all, or why does he go on writing? He must know by this time that it is a part of the institution that his "untruthful and absurd trash" shall be distilled into mine at the rate of about 3⅛ pages of the first to one column of the second. Your readers will never know how much they gain by the process, until Mr. James Smith publishes it all in a big book, or until they get hold of what he has already published. I have six pounds avoirdupois of pamphlets and letters; and there is more than half a pound of letters written to you in the last two months. Your compositor must feel aggrieved by the rejection of these clearly written documents, without erasures, and on one side only. Your correspondent has all the makings of a good contributor, except knowledge of his subject and sense to get it. He is, in fact, only a mask: of whom the fox

> O quanta species, inquit, cerebrum non habet.

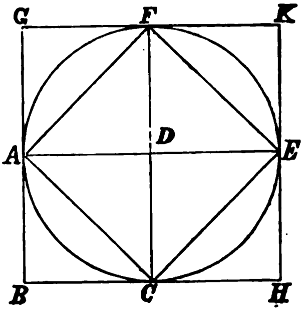
I do not despair of Mr. Smith on any question which does not involve that unfortunate two-stick wicket at which he persists in bowling. He has published many papers; he has forwarded them to mathematicians: and he cannot get answers; perhaps not even readers. Does he think that he would get more notice if you were to print him in your journal? Who would study his columns? Not the mathematician, we know; and he knows. Would others? His balls are aimed too wide to be blocked by any one who is near the wicket. He has long ceased to be worth the answer which a new invader may get. Rowan Hamilton, years ago, completely knocked him over; and he has never attempted to point out any error in the short and easy method by which that powerful investigator condescended to show that, be right who may, he must be wrong. There are some persons who feel inclined to think that Mr. Smith should be argued with: let those persons understand that he has been argued with, refuted, and has never attempted to stick a pen into the refutation. He stated that it was a remarkable paradox, easily explicable: and that is all. After this evasion, Mr. James Smith is below the necessity of being told he is unworthy of answer. His friends complain that I do nothing but *chaff* him. Absurd! I winnow him; and if nothing but chaff results, whose fault is that? I am usefully employed; for he is the type of a class which ought to be known, and which I have done much to make known.

A. DE MORGAN.

APPENDIX E.

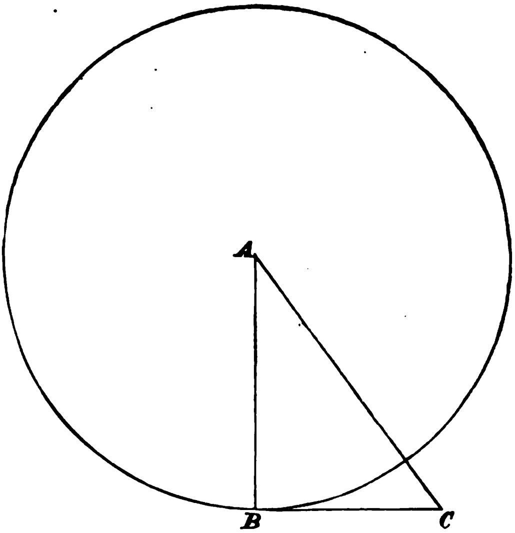
"TRUTH IS STRANGER THAN FICTION."

FIG 1



The Diagram (Fig. I.) is a facsimile of that in the address to the Reader. The area of the inscribed square A C E F is a mean proportional between the area of the circumscribing square G B H K and the area of the square A B C D on the radius of the circle; and it follows, that $4\left(\frac{\pi}{2}\right)^2 = \pi^2$, whatever be the value of π. Hence: The value of π cannot be an indeterminate arithmetical quantity. **These facts I have proved in the short address to the Reader.**

FIG. 2.



The Diagram (Fig. 2.) is a fac-simile of that on p. 367; that is to say, in the right-angled triangle ABC, the sides A B and B C, which contain the right angle B, are in the ratio of 4. to 3, and A B is the radius of the circle, by construction.

Now, no "*quantity of turning*," of the triangle A B C, round the angle point A, in either direction, could ever carry the angle B outside the circle, or bring the angle C within it.*

These facts are not self-evident to the "*We-We of the 'Athenæum;*'" or, in other words, they are not self-evident to the "*recognised*" PHILOMATHS "*of our time*," but they will be self-evident to every first class school-boy of a future generation.

My numerous correspondents are "*knocked over*" by one another most amusingly. Mr. Henry W. Toller of Stoney-gate House, Leicester, and others, charge me with making radii of the same circle *unequal*. The Rev. Professor Whitworth, Fellow of St. John's College, Cambridge, and Professor of Mathematics in Queen's College, Liverpool, charges me with making a certain chord and its subtending arc *equal:* or, put in other words, this learned gentleman charges me with making the perimeter of a certain polygon and the circumference of its circumscribing circle *equal.* The Rev. Geo. B. Gibbons, and others, charge me with making the perimeter of a regular polygon *greater* than the circumference of its circumscribing circle. How can all these charges be true? If I make the perimeter of any regular polygon *greater* than the circumference of its circumscribing circle, how, in the name of common sense, can I make a certain chord and its subtending arc *equal,* or radii of the same circle *unequal?* Are not the sides of a regular six-sided polygon, equal to the radius of a circumscribing circle? Is not the perimeter of a regular six-sided polygon, equal to six times the radius of its circumscribing circle?

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* I venture to put the following question to Mr. J. M. Wilson, Mathematical Master of Rugby School, *Senior Wrangler at Cambridge in* 1859, and author of a Treatise on Elementary Geometry, recently published. Have I *misapplied* the expression "*quantity of turning*" introduced by *you*, for the first time, into a text-book on Geometry?

Not one of my correspondents has ever attempted to *prove* his charges, *by logical reasoning upon sound and indisputable data.*

Letters of mine, on the *Quadrature of the Circle,* appeared in the *Correspondent* of Aug. 19 and 26, and Sept. 2, 1865, and the following Letter to the Editor appeared in that Journal of Sept. 9, 1865:—

SIR,

As Mr. Smith wishes the public to accept the fact which he believes he has proved—viz., that the true value of $\pi = \frac{25}{8}$, may I be permitted to ask him a question which seems to bear very closely on the subject?

Supposing the diameter of a circle to be 1 foot, what is the perimeter of an inscribed regular polygon having 18 sides?

In my attempts to solve this question, I have arrived at a result which seems worthy of notice.

Now, the side of the polygon subtends an angle of 20°, and if we denote the side by a, we have at once (the radius being r)—

$$\tfrac{1}{2} a = r \text{ sin. of } 10^\circ,$$

Or, since $r = \frac{1}{2}$,

$$\tfrac{1}{2} a = \text{sin. of } 10^\circ.$$

The value of this sine is given in "Hutton's Logarithmic Tables" as = ·1736482. Multiply by 18, and we get the perimeter of the figure = 3·1256676 feet.

If this be correct, and Mr. Smith be also correct, it follows that the circumference of the circle (which he makes to be 3·125 feet) is *less* than the perimeter of the regular polygon which it circumscribes. My only assumption is that of the value of the sin. of 10°. It is for Mr. Smith to say whether this value is incorrect or not, and, if incorrect, it is for him to set it right.

But my chief object is just to point out that we need not *theorize* about the matter at all. A plain practical man who does not understand mathematics, but who can just draw a diagram, may make a regular polygon of 18 sides for himself, and can tell by measurement that the perimeter, or whole way round it, is very nearly = $3\frac{1}{8}$ of the diameter, or breadth across; if anything, the proportion is a little greater than $3\frac{1}{8}$. Here is a simple practical test for the general public to judge by—as I suppose it to stand to common sense that the circumference of a circle on the same diameter is *greater* than the perimeter of the 18-sided figure. But Mr. Smith would, apparently, make it *equal* or *less*.

Offering this test for the use of any one interested in the question,

I remain, Sir,
Yours very truly,
WALTER W. SKEAT.

The writer of this Letter has made the fairest attempt that has ever come under my observation, to give a "*refutation*" of the truth of the geometrical theorem, that 8 circumferences of a circle are exactly equal to 25 diameters (his mechanical test excepted), which makes $\frac{25}{8} = 3·125$, the arithmetical value of the circumference of a circle of diameter unity. Mr. Skeat is an occasional contributor to the *Athenæum*, dates from Cambridge, and I have reason to believe, that in Cambridge is well known, as a "*recognised Mathematician.*"

It will be observed, that Mr. Skeat adopts as a premiss, a circle of diameter 1 foot; and it cannot be disputed, that the number of feet contained in the circumference of a circle, of which the diameter = 1 foot, is the thing to be ascertained. Todhunter—and he is one of the "*Men of the Time*"—in his work on Plane Trigonometry, observes :—" *The symbol* π *is invariably used to denote the ratio of the circumference of a circle to its diameter; hence, if* r *denote the radius of a circle, its circumference is* $2\pi(r)$." Does it not follow, that the symbol π denotes the circumference of a circle of diameter unity? This cannot be disputed. Well, then, it is self-evident—whether the length of the diameter of a circle be denoted by 1 foot, 1 yard, 1 mile, or 1 anything else—that the number of units of length contained in the circumference, is the arithmetical value of the symbol π.

Now, Mr. Skeat puts the following question :—" *Supposing the diameter of a circle to be* 1 *foot, what is the perimeter of an inscribed regular polygon having* 18 *sides?*" He then reasons thus :—"*Now, the side of the polygon subtends an angle of* 20°." Granted. " *And if we denote the side by* a, *we have at once (the radius being* r)—

$$\tfrac{1}{2}a = r \text{ Sin. of } 10^\circ$$

$$\textit{or, since } r = \tfrac{1}{2}$$

$$\tfrac{1}{2}a = \text{Sin. of } 10^\circ.$$

The value of this Sin. is given in Hutton's Logarithmic Tables as = ·1736482." Mr. Skeat then says :—" *Multiply by* 18, *and we get the perimeter of the figure* = 3·1256676 *feet.*" He then draws the following conclusion :—" *If this be correct, and Mr. Smith be also correct, it follows that the circumference of the circle (which he makes to be* 3·125 *feet) is less than the perimeter of the regular polygon which it circumscribes.*"

There are more fallacies than one in this piece of reasoning. Mr. Skeat, in the first place, falls into the same blunder as my correspondents, the Rev. Professor Whitworth and the Rev.

Geo. B. Gibbons; that is to say, he assumes (unwittingly no doubt) that the *trigonometrical functions* of angles are lengths, and not ratios of one length to another. But, he falls into another blunder. He assumes (unwittingly no doubt) that the natural sine and trigonometrical sine are *arithmetically* the same in all angles. (In this he falls into the same blunder as Mr. Alex. Edward Miller of Lincoln's Inn, another contributor to the *Correspondent.*) * This is true of certain angles only. It is true of an angle of 30°. Both the *natural* and *trigonometrical* sine of an angle of 30° is ·5, and is equal to the radius of a circle of diameter unity; or half the side of a regular inscribed hexagon to a circle of radius = 1. Is not the sine of an arc, half the chord of twice the arc? This fact, the Rev. Geo. B. Gibbons not only admits, but adopts as a *premiss*. Now, according to the *reasoning* of Mr. Skeat, ·⅙ = ·166666 with 6 to infinity, should be the natural sine of an angle of 10°, which, according to Hutton, is = ·1736482: and, according to the reasoning of Mr. Skeat, ·¼ = ·25 should be the natural sine of an angle of 15°, which, according to Hutton, is = ·2588190. Now, whether Hutton's Logarithmic Tables are fallacious, or not fallacious, Mr. Skeat's *reasoning* cannot be sound.

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* Mr. Miller, though not a *professional*, is a "*recognised Mathematician.*" In a Letter of his which appeared in the *Correspondent* of January 6, 1866, he says:—"Mr. Smith's remarks scarcely deserve a reply, and I only offer one lest some non-mathematical reader should think they do not admit of one. 'There is no distinction between the trigonometrical and natural sines of angles.' Trigonometry may be defined as 'the art of using the geometrical ratios,' of which I need not say that the sine is the principle. Mr. Smith is, I suppose, referring to the distinction between the natural and *logarithmic* sines, &c., which I presume he has seen in the Tables, without quite comprehending, the one being merely the arithmetical values of the different sines, the other the logarithms of those values." Making the trigonometrical and natural sines *arithmetically* the same in all angles, is the grand stumbling-block of all my opponents.

The circumference of a circle of radius $= \frac{1}{2} = \pi$, whatever be the value of π; and since the circumferences of circles are to each other as their radii, it follows, that the circumference of a circle of radius $= 1 = 2\pi$, whatever be the value of π.

Now, Mr. Skeat, in his Letter to the *Correspondent*, frankly makes the following admission; and I know from experience, that it is a *rare* thing for a Mathematician to admit anything in a mathematical controversy:—"*My only assumption is that of the value of the Sin. of* 10°," and he admits that he takes this from Hutton's Logarithmic Tables, in which, it cannot be disputed, that the Sin. of an angle of 10° is given as = ·1736482: and it cannot be disputed, that this, multiplied by 18 = 3·1256676, and is *greater* than $\frac{25}{8} = 3·125$. But, it follows, if Mr. Skeat be "*correct*," and Hutton be also "*correct*," that the Sin. of an angle of 20° multiplied by 18, should be greater than 2 (3·125) = 6·25. Now, Hutton gives the Sin. of an angle of 20° as = ·3420201. Multiply by 18, and we get 6·1563618, which is *less* than 6·25. Well, then, it is self-evident that Mr. Skeat and Hutton cannot both be "*correct*," and both may be wrong. I maintain that both are wrong; or in other words, I maintain that the *reasoning* of Mr. Skeat, and Hutton's Logarithmic Tables, are both *fallacious*. Let some of the *recognised* PHILOMATHS "*of our time*" prove, that I make the perimeter of any regular polygon *greater* than the circumference of its circumscribing circle. I challenge the PHILOMATHS of the day, to furnish the proof. They cannot do it, *NO*, not even by assuming the *infallibility* of Logarithmic Tables, which it may be admitted, "*have been calculated by experts in every country in Europe.*"

The arithmetical value assigned to π by "*recognised Mathematicians*" (the Rev. Professor Whitworth excepted) is 3·14159265, &c., and is neither "*finite*" nor "*determinate*," whatever the Rev. Professor Whitworth may say. Does not this make π = 3·14159265, with a never-ending string of decimals? Does not the series "π =

$\frac{16}{\sqrt{3}}\left(\frac{1}{1\cdot3} + \frac{2}{5\cdot7}\left(\frac{1}{9}\right) + \frac{3}{9\cdot11}\left(\frac{1}{9}\right)^2 + \frac{4}{13\cdot15}\left(\frac{1}{9}\right)^3 + \text{\&c.,}\right.$" work out to this result? Let that "*recognised*" PHILOMATH, De Morgan, "*brush away*" the &c., and prove, that $4\left(\frac{3\cdot14159265}{2}\right)^2$ is not equal to π^2, when 3·14159265 is *assumed* to be the arithmetical value of the symbol π. Having "*brushed away*" the &c., we may next "*brush away*" the last decimal, then the next, then the next, and so on, till we have "*brushed away*" all the decimals. At every step we get the equation, $4\left(\frac{\pi}{2}\right)^2 = \pi^2$; that is to say, whether we *assume* π = 3·14159265, 3·1415926, 3·141592, 3·14159, 3·1415, 3·141, 3·14, 3·1, or, π = 3, and work out the calculations, we get the equation $4\left(\frac{\pi}{2}\right)^2 = \pi^2$, and we cannot alter this result by adding fifty or any other number of decimals to 3·14159265, after we have "*brushed away*" the &c. How, then, in the name of common sense, can π be an indeterminate arithmetical quantity? Can any "*recognised* PHILOMATH" show us how to divide &c. by 2 and square it? Let De Morgan try! If he do, he will most assuredly "*winnow*" himself, more than he has ever *winnowed* James Smith: "*and if nothing but chaff results*" whose fault will that be?

The Reader of the Article: PSEUDOMATH, PHILOMATH, GRAPHOMATH: which appeared in the *Athenæum* of January 4, 1868 (*See Appendix D*), will observe, that the writer, who gives his name, says of Mr. James Smith:—"*Surely the cylometer* (*Query—cyclometer*) *is a Darwinite development of a spider, who is always at circles, and always begins again when his web is brushed away.*" That "*recognised* PHILOMATH" has himself informed us in one of his Budgets of Paradoxes, that he has made himself "*a public scavenger of science*," and "*looks down upon other scavengers*," such as "Montucla, Hutton, &c., *as mere historical drudges*," and "*not fit to compete*" with him. This may be true, and he may be able to "*brush away*" the *fantastical* webs of all the "*recognised*" mathematical spiders in the world.

It is possible, he may succeed in "*brushing away*" some of his own *fantastical* webs, and then make an attempt to "*brush away*" the latest web spun by the *non-recognised* "*spider.*" If so, I venture to tell that "*recognised* PHILOMATH," that his efforts will most assuredly result in "*miscalculated and disorganised failure.*"

In his Budget of Paradoxes, No. 27 (*See Athenæum*, July 8, 1865), Professor De Morgan says:—"*He* (Mr. Smith) *does not know that Sines as well as* π *are interminable decimals, of which the tables, to save printing, only take in a finite number.*" Mr. Smith *knows* that the sine of an angle of 30° is ·5. Is not ·5 a terminable decimal? Where is the mathematician who will venture to dispute it? Well, then, is not the learned Professor "*knocked over*" by himself? If not, surely that "*public scavenger of science*" must "*brush away*" his "*interminable decimal*" web, before he can hope to "*brush away*" the last spun web of the "*Darwinite development of a spider.*" Would he not be more "*usefully employed*" in endeavouring to "*brush away*" the *finite and determinate* web of that "*recognised*" mathematical spider, the Rev. Professor Whitworth, than in doing so "*much to make known,*" that "*Darwinite development of a spider,*" James Smith, Esq., of Liverpool?

In his correspondence with the Rev. Professor Whitworth, James Smith has proved, not only that a circle and a square of *equal* superficial area, may, and do exist; but he has proved, that they may be geometrically isolated and exhibited, in a variety of ways: and it follows, that a definite relation exists between the diameter and circumference of a circle, and is *arithmetically* expressible with perfect accuracy.

It is asserted in the article: PSEUDOMATH, PHILOMATH, GRAPHOMATH: that "*Rowan Hamilton, years ago, completely knocked him* (Mr. James Smith) *over:*" and after stating that: "*There are some persons who feel inclined to think that Mr. Smith should be argued with:*" it is asserted, "*let those persons understand* *that he hos been argued with, refuted, and has never attempted to stick a pen into the refutation.*" I shall leave it to Readers to form their own opinion as to the *truth* of these ASSERTIONS. The "*recognised* PHILOMATH," De Morgan, may be "*a lover of learning,*" but I am afraid it can hardly be said of him :—He is a lover of TRUTH.

I was brought into contact with the late Sir. Wm. Rowan Hamilton in 1859, at the Meeting of the "*British Association,*" at Aberdeen. My correspondence with him was a very short one. I believe I only wrote him three Letters, and certainly only received two communications from him. It is true that I presented him with two of my pamphlets ; and on two or three occasions, I had conversations with him on the subject of the ratio of diameter to circumference in a circle. Our correspondence closed with the following communications :—

OBSERVATORY, NEAR DUBLIN,
November 11, 1861.

Sir W. R. Hamilton presents his compliments to James Smith, Esquire, of Barkeley House, Seaforth, near Liverpool.

Sir W. R. H. had conceived that the correspondence between Mr. Smith and himself was closed, by his letter of the 27th April last ; but he remembers perfectly the fact of his lately meeting Mr. Smith on the steps of one of the public buildings in Manchester.

He has since sought to fulfil his promise, then in politeness given, at Mr. Smith's request, that he would read part, at least, of the last paper by Mr. Smith, on the Quadrature of the Circle.

He has accordingly done so, to quite a sufficient extent, and with quite sufficient care, to be satisfied that there would be no use in his reading any further.

Sir W. R. H. regrets to say that he does not consider Mr. Smith to understand the principles of *decimal arithmetic* as applied to *infinite series.* And the fallacies, hence resulting, appear to him to vitiate the whole of Mr. Smith's arithmetical argument.

As regards the geometrical theorem, which has been known to

Mathematicians for about two thousand years, that *eight circumferences of a circle exceed twenty-five diameters*, it has been recently confirmed by Sir W. R. H. in an elementary demonstration, which he may perhaps be induced to republish.

But as it must be obviously useless to continue, or rather to reopen this correspondence, Sir W. R. H. hopes that Mr. J. Smith will not consider him as discourteous, if he shall not in future acknowledge any printed or other communication on the subject.

JAMES SMITH, ESQ.,
Barkeley House, Seaforth, near Liverpool.

BARKELEY HOUSE, SEAFORTH,
NEAR LIVERPOOL, 15*th November*, 1861.

Mr. James Smith presents his compliments to Sir William Rowan Hamilton, LL.D., &c., &c., of the Observatory, near Dublin, and "The Astronomer Royal of Ireland," and begs to apologize for the delay in acknowledging the receipt of Sir W. R. H.'s note, for which he could give a good reason.

Mr. Smith observes the distinction drawn by Sir W. R. H. between meeting Mr. S. *at the British Association* in Manchester, and meeting him *on the steps* of one of the public buildings of Manchester, the public building in question being the one in which the Physical Section of the Association held its meetings on that occasion; and Mr. Smith regrets to find that Sir W. R. H., who, he had been led to believe, was the most polite and polished gentleman in Ireland, can, when it is convenient to do so, play a part greatly at variance with his general character.

Mr. Smith could prove, if necessary, through the gentleman to whom Sir W. R. H. appealed, to evidence the fact of his disbelief in Mr. Smith's theory, that Sir W. R. H. did promise to give the Letter Mr. S. had addressed to the President and Committee of the Physical Section of the British Association a careful perusal; and Mr. S. regrets to find that the politeness of Sir W. R. H. on that occasion, would appear to have been a mere cloak to conceal his want of candour.

Mr. Smith observes that Sir W. R. H. considers a knowledge "*of decimal arithmetic as applied to infinite series*" essential, for the purpose of demonstrating the "*geometrical theorem*" of how many times the diameter of a circle is contained in its circumference; and Mr. S. cannot but express his surprise that Sir W. R. H. should be so inconsistent, as to profess to recognise, and believe in, the "*authority*" of the Mathematicians of two thousand years ago, who certainly had no knowledge whatever of "*decimal arithmetic as applied to infinite series.*"

Mr. Smith would not have again intruded himself upon Sir W. R. H., if he had not had new matter to communicate, which he thought was well deserving of consideration by the highest "*authorities*" in Mathematical Philosophy. To Mr. S. it is a matter of indifference that Sir W. R. H. declines to re-open a correspondence with him, and would only remark that Sir W. R. H. may yet live to acknowledge the fact, that "*the wisdom of the wise may be destroyed, and the understanding of the prudent brought to nothing.*"

Mr. Smith has only to remark in conclusion, that he will spare Sir W. R. H. the pain of being either discourteous or uncandid for the future, so far as he is concerned. It is quite sufficient for Mr. S., for the convenience of regulating his future course of procedure, to have discovered, that Sir W. R. H. can be both discourteous and uncandid, when it suits his purpose to display those qualities.

SIR W. ROWAN HAMILTON, LL.D., &c.,
Observatory, near Dublin.

These communications appear in a pamphlet which I published in 1865. I sent a copy of this pamphlet to Sir Wm. Rowan Hamilton, during his life time. I quote the following from that pamphlet, which appears as a foot-note on the same page as Sir William's final communication.

"The reader will observe, that Sir William Rowan Hamilton considers, that the '*geometrical theorem*' of how many times the diameter of a circle is contained in its circumference, can only be solved by one who has a knowledge of '*decimal arithmetic as applied to infinite series.*' To establish this '*assumption,*' Sir William must prove the following algebraical formula to be false, which he will find to be an impossibility. If the diameter of a circle be 12, and $3 (sr \times \pi) = x$, then, $\frac{r^2 \times \pi}{4} = x$; therefore, $x =$ the difference between the area of a circle of which the diameter is 8, and the area of a circle of which the diameter is 10; whatever be the arithmetical value of π. How truly has Professor de Morgan told us, that '*crammed erudition does not cast out any hooks for more.*' Poor Sir William! How unfortunate! What a pity he should have to make up his mind, like my correspondent, Lieut.-General T. Perronet Thompson, to be looked upon in the future by every first class school-boy, as a mere simpleton, notwithstanding the profundity of his '*mathematical wisdom.*'"

With reference to this quotation, I may put the question:—Will any "*recognised Mathematician*" venture to tell me, that $3 (s\, r \times \pi)$ is not equal to $\frac{r^2 \times \pi}{4}$, when $s\, r$ denotes the semi-radius of a circle, r the radius, (whatever be the value of π), and the diameter of the circle $= 12$? It may be admitted, that this is a "*particular case,*" and "*quite unique:*" or, in other words, it may be admitted, to be true only of a circle of which the diameter $= 12$, that the equation $3 (s r \times \pi) = \frac{r^2 \times \pi}{4} =$ the difference between the areas of circles of diameter $= 8$, and diameter $= 10$.

Well, then, I put the following questions to the "*recognised*" PHILOMATHS "*of our time.*" With reference to the diagram (Fig. 2) on page 394, Is not $\pi (r^2) =$ area in every circle? Is not $\pi (A B^2) = \frac{\pi}{4}$, when A B the radius of the circle $= \frac{1}{2}$? Is not $12{\cdot}5\left(\frac{\pi}{50}\right) = \frac{\pi}{4} =$ whatever be the value of π? Is not $\frac{\pi}{4}$ *equal* to the area of a circle of diameter unity, whatever be the value of π? Is not the area of any circle divided by $\frac{1}{4} (\pi)$ equal to the area of a circumscribing square to the circle, whatever be the value of π? Is not the property of one circle the property of all circles?

Does it not follow that $\frac{\text{Area of circle}}{\frac{1}{4}(\pi)}$ = area of a circumscribing square to every circle, whatever be the area of the circle, or, whatever be the value of π? When A B = 4, does not B C = 3, and A C = 5? When A B and B C, which contain the right angle B, are in the ratio of 4 to 3, is not $(AB^2 + BC^2 + AC^2) = 3\frac{1}{8}(AB^2)$? Is not $12 : 4(3\frac{1}{8}) :: 3 : 3\frac{1}{8}$? Is not $4\pi(s\,r^2) = \pi(r^2)$? To all these questions—and I might put others—every HONEST Mathematician must give an answer in the affirmative.

Reader, mark what follows. $\frac{\text{Area}}{\frac{1}{4}(\pi)}$ = area of a circumscribing square to every circle. Side of every square = diameter of an inscribed circle. $\frac{\text{Diameter}}{2}$ = radius in every circle; and it follows, that $\frac{\text{Area}}{\frac{1}{4}(\pi)} = 4(r^2)$ in every circle.

Well, then, let the area of a circle be any given finite quantity, say 60. Then: $\frac{60}{\frac{1}{4}\pi}$ = area of a circumscribing square to the circle. Can any "*recognised Mathematician*" find the area of a circumscribing square to a circle of area = 60? Not with any π of indeterminate arithmetical value, certainly! A Mathematician of the Wilson and Whitworth stamp, would probably tell me, that the area of a circle is an indeterminate arithmetical quantity, whether the diameter be *arithmetically* determinate or *arithmetically* indeterminate. (One of my correspondents told me—and he was a first class wrangler of his year—that if the diameter of a circle be finite, away goes circumference and area into decimals without end: and if the circumference of a circle be finite, away goes diameter and area into decimals without end.) For the sake of argument let this be granted. Then, it would follow, of necessity, that the area of no square could be arithmetically determinate.

Well, then, let A = 60, and denote the area of a circle, Then: $\frac{A}{\frac{1}{4}\pi} = \frac{60}{\frac{1}{4}(3\frac{1}{8})} = \frac{60}{\cdot78125} = 76\cdot8$ = area of a circumscribing square to the circle. $\sqrt{76\cdot8}$ = diameter of the circle. $\frac{\text{Diameter}}{2} = \frac{\sqrt{76\cdot8}}{2} = \sqrt{19\cdot2}$ = radius of the circle. And, $4(r^2)$ $= 4(\sqrt{19\cdot2})^2 = 4 \times 19\cdot2 = 76\cdot8$ = area of a circumscribing square to the circle, when the area of the circle = 60; and demonstrates that 8 circumferences = 25 diameters in every circle, which makes $\frac{25}{8} = 3\cdot125$, the true arithmetical value of the symbol π. Can any man fail to perceive, that to "*upset*" this conclusion, he must prove that the area of a circumscribing square to a circle of diameter 1 mile is either greater or less than 1 square mile? Can we not give the superficial area of 1 square mile, either in feet, yards, or inches, with arithmetical accuracy?

Whenever a Philomath shall arise, who can prove, that under no circumstances can the area of a circumscribing square to a circle be a determinate arithmetical quantity, he will "*knock over*" James Smith, and nail him to the "*barn-door*" as "*the delegate of miscalculated and disorganized failure.*"*

JAMES SMITH.

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* See, *Athenæum*, July, 25, 1868: Article: "Our Library Table."

ERRATA.

Page

44. 13th line from top. For *lines*—read, *line*.

45. 6th line from top. Omit the words: *and join B D and C D*.

8th line from top. For *circle X Y*—read, *circle X Z*.

46. 3rd line from top. For *therefore*, $A D^2$—read, *therefore*, $A D$.

16th line from bottom. For *squares*—read, *square*.

48. 6th line from top. For (3125 × 25)—read, (3·125 × 25).

63. 13th line from bottom. For *capacities*—read, *capacity*.

70. 7th line from top. For *O V X*—read, *O V R*.

91. 14th line from bottom. After the words, *and join H C*—insert, *From the point C draw a straight line at right angles to AC, to meet and terminate in the circumference of the circle Y at the point G.*

92. 16th line from bottom. For *and E D*—read, *and the square of E D*.

100. 8th line from top. For (12·5 + 23·84185791015625)– read (12·5 × 23·84185791015625).

106. 6th line from bottom. For *indenty*, read *identity*.

108. 8th line from bottom. For *indentity*—read, *identity*.

109. 12th line from bottom For *indentity*—read, *identity*.

119. 12th line from top. For ·62—read, ·625.

121 7th line from bottom. For *circle X Y Z*—read, *circle X or Z*.

122. 5th line from top. For $4 \pi (E K)$—read, $4 \pi (E K^2)$.

133. 15th line from top. For *two*—read, *four*.

144. 11th line from top. For *circle X*—read, *circle Z*.

155. 14th line from top. For (3317·76 — 51·125) read, (3317·75 — 51·1225).

165. 2nd and 3rd lines from top. For 3317·76 — 51·4 = 3265·94—read, 3317·76 — 51·84 = 3265·92. And, for $\sqrt{3265{\cdot}94}$—read, $\sqrt{3265{\cdot}92}$.

173. 10th line from bottom. For (*A B* — *E B*), read, (*A B* — *A E*).

174. 17th line from bottom. For *F A*—read, *F C*.

181. 15th line from top. For $\frac{E\,B}{F\,E}$—read, $\frac{F\,B}{F\,E}$.

3rd line from bottom. For *Sin. of angle E B F*—read, *Sin. of angle E F B*.

182. 17th line from top. For *Sin. of angle E B F*—read, *Sin of angle E F B*.

4th line from bottom. For $\frac{B\,A}{E\,A}$—read, $\frac{B\,A}{F\,A}$.

183. 11th line from top. For *Log.* 9·7781510—read, *Log.* 9·7781513.

12th line from bottom. For $\sqrt{(399{\cdot}9^2 + 532{\cdot}8\ )}$—read, $\sqrt{(399{\cdot}6^2 + 532{\cdot}8^2)}$.

Bottom line. For 396·57—read, 399·57.

226. 2nd line from top. For *N n K*—read, *M n K*.

9th line from top. For *D R*—read, *A R*.

234. 11th line from top. For 1 *to* 4—read, 1 *to* 3; and then insert: *which makes A C to A B in the ratio of* 1 *to* 4; or, for *C B*—read, *A B*.

235. Top line. For $(D B^2 \dot{-} y B)$—read, $(D B^2\ y B^2)$.

237. 6th line from bottom. For *D A*—read, *O A*.

With reference to these errata, every *reasoning geometrical* reader will infer, and rightly infer, that most of them are slips, and mere slips, of the compositor. I have not thought it necessary to read for errata beyond the Letter to Professor Stokes. There may be other errors, but they will be such as any honest "*reasoning geometrical investigator*" will readily detect, and at once correct. Failing health prevented me from reading for press, with the care that, otherwise, I might and should have done.

J. S.